MAESTRO-12, 2020, **Holomorphic dynamics, fractals, thermodynamic formalism** *(short description)*, by Feliks Przytycki

The project is organized in six inter-connected tasks/topics, organized into 18 Objectives, which are either concrete problems to be solved or/and fields to be investigated. Before presenting them we introduce some basic notions and ideas.

0) Introduction, basic notions

– *Iteration of rational maps.* The main landscape of the proposed investigations is the Riemann sphere with a holomorphic function acting on its open domain and limit sets of this action. For f rational of degree at least 2 one considers Julia set $J(f)$, with chaotic dynamics, and its complement, called Fatou set, by definition the domain of normality of iterates f^n. It has finite number of periodic components and its f^{-n} preimages. These periodic ones are immediate basins of attraction to attracting periodic orbits inside or parabolic periodic orbits in their boundaries or *singular domains with elliptic dynamics* (irrational rotation in appropriate holomorphic coordinates): Siegel discs or Herman rings. Julia set can be defined also as a limit set of preimages of non-exceptional points, or closure of repelling periodic orbits. For polynomials it is boundary of the basin of attraction to ∞. Considering quadratic polynomials $f_c(z) = z^2 + c$ we see that the set $J(f_0)$ is the unit circle, wiggly for $c \neq 0$ but close to 0, subject to self-glueings resulting with Mandelbrot's basilica or Douady's zoo: rabbit, dragons etc. When $f_c^n(0) \to \infty$ i. e. c is not in Mandelbrot set, then $J(f_c)$ ceases to be connected and becomes a Cantor dust. For rational maps Julia set can be also the entire sphere or e.g. of Sierpiński carpet type. See textbooks by J. Milnor or L. Carleson and T. Gamelin.

– *Thermodynamic formalism.* It is a powerful and inspiring method taken from equilibrium statistical physics to study the limit sets. Among founders of it are Ya. Sinai, R. Bowen and David Ruelle. He wrote in [Ruelle]: "thermodynamic formalism has been developed since G. W. Gibbs to describe [...] physical systems consisting of a large number of subunits". In particular one considers a *configuration space* Ξ of functions $\mathbb{Z}^n \to \mathbb{A}$ on the lattice $\mathbb{Z}^n$ with interacting values in $\mathbb{A}$ over its sites, e.g. "spin" values in the Ising model of ferromagnetism. One considers probability distributions on Ξ, invariant under translation, called *equilibrium states* for potential functions on Ω generated by the interactions. Given a mapping $f : X \to X$ one considers as a configuration space the set of trajectories $n \mapsto (f^n(x))_{n\in\mathbb{Z}_+}$ or $n \mapsto \Phi(f^n(x))_{n\in\mathbb{Z}_+}$ for a test function $\Phi : X \to Y$.

– *Geometric pressure and Hausdorff dimension.* Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be rational map of degree $\geq$ 2and $X \subset J(f)$ be compact f-invariant. Define $\boxed{P_{\rm var}(f,\phi) = \sup_{\mu\in\mathcal{M}(f)} \left(h_\mu(f) + \int_X \phi\, d\mu\right)}$ here for $\phi = \phi_t := -t\log|f'|$ with $t \in \mathbb{R}$, where $\mathcal{M}(f) = \mathcal{M}(f, X)$ is the set of all f-invariant Borel probability measures on X and $h_\mu(f)$ is measure theoretical entropy. Every μ for which $P_{\rm var}(f, \phi)$ is attained is called *equilibrium state* or *measure*. $\boxed{P(t) := P_{\rm tree}(f,\phi) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{f^n(x)=z,\, x\in J(f)} |(f^n)'(x)|^{-t}}$ is called *geometric pressure*. For $X = J(f)$ it does not depend on z except z fast accumulated by critical trajectories, of Hausdorff dimension 0, and coincides with $P_{\rm var}(f, \phi)$. If f is expanding (i.e. $|(f^n)'(x)| \geq \lambda > 1$ for n large enough and all $x \in X$) and open on f-invariant $X \subset J(f)$ then if $P(t_0) = 0$ for $t = t_0$, we can take a small disc around z and distribute a measure μ_{t_0} on $J(f)$ roughly according to $\mathrm{diam}(\mathrm{Comp}_x\, f^{-n}B(z,r))^{t_0}$, i.e. according to $|(f^n)'(x)|^{-t_0}$, provided *distortion is bounded* i.e. no critical points interfere. This μ_{t_0} happens to be an equilibrium (Gibbs) state for ϕ_{t_0}. We obtain $\boxed{\mu_{t_0}(B) \approx (\mathrm{diam}\, B)^{t_0}}$. Hence Hausdorff dimension $\boxed{\mathrm{HD}(J(f)) = t_0}$ (Bowen-Manning's formula).

We define in dimension 1 *hyperbolic dimension* $\mathrm{HD}_{\rm hyp}(X)$ for an arbitrary f-invariant set $X \subset \overline{\mathbb{C}}$ as supremum of Hausdorff dimensions of f-invariant hyperbolic (i.e. expanding) repellers $Y \subset X$. Then for e.g. $X = J(f)$ we have $\mathrm{HD}_{\rm hyp}(J(f))$ being the first zero of $t \mapsto P_{\rm var}(f, -t\log|f'|$, which is a generalization of Bowen-Manning's formula above, see [PUbook].

– *Non-uniform hyperbolicity in strong sense,* in dimension 1, e.g. for rational f means: There exists $C > 0$ such that Lyapunov exponent $\chi_\mu(f) := \int \log|f'|\, d\mu$ is $\geq C$ for every ergodic $\mu \in \mathcal{M}(f, J(f))$. Note that by Birkhoff Ergodic Theorem $\chi_\mu(f) = \lim_{n\to\infty} \frac{1}{n} \log|(f^n)'(x)|$ for μ-a.e. x. See e.g. [Prz:ICM-18] and [PRS:03]. Equivalent is *Backward exponential shrinking*, saying that for constants $r > 0, 0 < \lambda < 1$ and all n large enough, for all $x \in J(f)$ and component B_n of $f^{-n}(B(x, r))$, $\mathrm{diam}\, B_n < \lambda^n$. It is called *Topological Collet-Eckmann, TCE,* due to a topological characterization. Stronger than TCE: *semi-hyperbolicity* says that all critical $c \in J(f)$ are non-recurrent. A non-uniform hyperbolicity, as yielding "reluctant recurrence" of c, allows to control distortion of B_n's.

1) Objectives and their description

Task 1. The dichotomy: fractal or analytic sets

• OBJECTIVE O1.1. Answer if the following is true in full generality, or excluding hardest situations: For every rational function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of degree at least 2, its Julia set $J(f)$ has Hausdorff dimension $\mathrm{HD}(J(f))$ bigger than 1 provided it is connected, except f is a finite Blaschke product in some holomorphic coordinates, or a two-to-one holomorphic factor of a Blaschke product.

This was proved in 80-ties by Anna Zdunik (in cooperation with the PI) for polynomials, where $J(f) = \mathrm{Fr}\,\Omega_\infty$, but since then remains open for general rational functions, see [Prz:ICM-18] where some later progress by the PI was outlined. Recently this was proved for all polynomials with $J(f)$ not totally disconnected, [PZ:20]. Zdunik's proof answered O1.1 in positive also for rational functions in presence of an attracting periodic orbit and simple-connected immediate basin Ω of its attraction. In non-Blaschke case, due to some large deviations, see O4.3, she constructed an iterated function system for iterates of f with the limit set $\Lambda \subset J(f)$ hyperbolic with $\mathrm{HD}(\Lambda) > 1$ Much later the PI proved that such Λ can be found in $\mathrm{Fr}\,\Omega$ so $\mathrm{HD}_{\mathrm{hyp}}(\mathrm{Fr}\,\Omega) > 1$ or f is special as above. See [Prz:ICM-18] with further references, e.g. [Prz:06].

Theorem above saying that $\mathrm{HD}_{\mathrm{hyp}}(\mathrm{Fr}\,\Omega) > 1$ has local character; it is sufficient that f is defined holomorphic only on a neighbourhood of $\mathrm{Fr}\,\Omega$ repelling to the side of Ω, called then RB-domain, see [PUZ:89], with analytic boundary in place of the circle for Blaschke product in its assertion.

In view of this, the situations remained to be understood to answer O1.1 in positive, have been where $\overline{\mathbb{C}} \setminus J(f)$ consists only of immediate parabolic basins (not looking hard) Siegel discs and Herman rings, together with all their pre-images for iterates of f.

For a Siegel disc S it is not sufficient to study only its boundary, since it can be smooth. Fortunately if it has Brjuno but not Herman (in particular not diophantine) rotation number the boundary $\mathrm{Fr}\,S$ is a Jordan curve "hairy" outside S and the hair being the (closed) postcritical set has Hausdorff dimension 2, [CDY]. In the case the only critical points in $J(f)$ are in $\mathrm{Fr}\,S$ (conjecturally with Herman rotation number) $J(f)$ resembles the limit set of an apollonian packing again with Hausdorff dimension > 1.

• OBJECTIVE O1.2. Develop the theory of *mean wiggliness* in relation with Hausdorff dimension. (in collaboration with J. Graczyk, P. Jones and N. Mihalache), related to Michel Herman's question, if for a.e. rotation number Siegel discs $\mathrm{HD}(\partial S) > 1$. See [BS].

Task 2. Geometric pressure, geometric equilibria, periodic orbits

• OBJECTIVE O2.1. A variant of geometric pressure is *periodic pressure* P_{Per} which is defined as P_{tree} in Introduction but with $x \in \mathrm{Per}_n$, periodic of period n, rather than f^n-preimages of z. Answer whether $P_{\mathrm{Per}}(t) = P_{\mathrm{tree}}(t)$ for all rational functions.

In [PRS:04], this was proved under an additional assumption H. See also [BMS] for a class of polynomials. Does Hypothesis H always hold? In particular can exponentially large bunches of periodic orbits exist exponentially close to a Cremer fixed point (non-linearizable, "hairy") ?

Many questions on geometric pressure concern also maps of interval, see [PrzRiv:19]; more precisely <u>*generalized multimodal maps of interval*</u> defined on a finite union $\widehat{I}$ of disjoint closed intervals to $\mathbb{R}$, of class C^3. The set K of points whose forward trajectories do not escape corresponds to complex Julia set where the (chaotic) dynamics is studied.

In both, the complex and real cases, the PI plans to continue developing inducing technique to understand e.g. phase transitions for the geometric pressure function $P(t)$ and equilibrium measures, called then *geometric equilibria*, [PrzRiv:11], [PrzRiv:19] and [CorRiv:19]. Allowing discontinuities for generalized multimodal maps as in [PrzRiv:19] seems also treatable. Sample objectives would be

• OBJECTIVE O2.2. Verify that in the theorems asserting probability laws such as Central Limit Theorem CLT and Law of Iterated Logarithm LIL, for TCE rational maps (or more general classes) and unique geometric equilibria, see [PrzRiv:11], the Lipschitz observables can be replaced by Hölder (or more general) ones. Prove Almost Sure Invariance Principle.

• OBJECTIVE O2.3. For TCE a rational map (or map of interval) study geometric equilibrium measures μ from the geometric measure theory point of view, e.g. its *Scenery flow*, defined for every $x \in \mathbb{R}^d$ as the flow of measures $\mu_{x,t} = \frac{\mu(e^{-t}\mu(A)+x)}{\mu(\overline{B}(x,e^{-t}))}$ for $A \subset \mathbb{R}^d$. See e.g. [Kaenmaki] and [BFU].

Task 3. Geometric coding trees (gct), accessibility, Lyapunov exponent

One can replace the coding of $\mathrm{Fr}\,\Omega$ by angles of radial limits for a Riemann mapping $R : \mathbb{D} \to \Omega$ in the Task 1, by coding due to a geometric coding tree, gct. Given a rational map f (or branched covering as in Task 6.) this is a graph in our space (here: Riemann sphere) built as follows:

Join a *root* point z_0 with its f-preimages by curves $\gamma_1, ..., \gamma_d$, next lift these curves by f^{-n}. Preimages of z_0 are declared: vertices, the lift curves are edges. This gives a tree graph $\mathscr{T}$ in which infinite paths (of edges and vertices) give a coding map $z_\infty : \{1, ..., d\}^{\mathbb{Z}_+} = \Sigma^d \to \Lambda$ being the set of limits of convergent paths. f on Λ is coded by the shift map $\varsigma : \Sigma^d \to \Sigma^d$.Clearly $z_\infty \circ \varsigma = f \circ z_\infty$. If $\mathscr{T}$ is in Ω then the convergence corresponds to the existence of radial limits (radii correspond to the coding sequences in Σ^d). The convergence holds for all sequences except a thin set in Σ^d, thin-to-one, by Beurling's Theorem, see [PSkrzy] for general gct's. If the map f extends holomorphically to a neighbourhood of the closure $\overline{\Lambda}$ in $\overline{\mathbb{C}}$, then Λ is called a *quasi-repeller*, see [PUZ:89].

• OBJECTIVE O3.1. Answer the following question. Is $\overline{\Lambda}$ equal to the accumulation set of all the vertices/edges? Does the assumption Λ is a quasi-repeller help? For trees in a basin of attraction Ω it is so, $\overline{\Lambda} = \mathrm{Fr}\,\Omega$ due to radial limits interpretation. No dynamics is needed.

Gct's allow to replace Riemann mapping in the absence of Ω, providing a considerably faithful coding of limit sets, allowing to transport to it Gibbs (equilibrium) measures from the shift space. See e.g. [PUZ:89] or [Prz:ICM-18] and recently [DPTUZ], see Task 5. Yet it still carries many mysteries to be investigated.

• OBJECTIVES O3.2-3 concern a) accessibility of points in $\mathrm{Fr}\,\Omega$ or Λ radial or along a branch in $\mathscr{T}$, of points with lower (upper) Lyapunov exponent $\boxed{\underline{\chi}(f,x) := \liminf_{n\to\infty}(\limsup)\frac{1}{n}\log|(f^n)'(x)|}$ positive, see [Prz:94, Prz:ICM-18], b) $\underline{\chi}(f, f(c)) \geq 0$ see [LPS], c) (im)possibility of x with $\overline{\chi}(f,x) > 0$ in the postcritical set for infinitely renormalizable polynomial (in collab. with G. Levin and W. Shen).

Related are my papers with K. Gelfert and M. Rams on Lyapunov spectra expressing Hausdorff dimension of the sets $\mathscr{L}(\alpha,\beta) := \{z : \underline{\chi}(z) = \alpha, \overline{\chi}(z) = \beta\}$ for all (non-exceptional) rational maps (and multimodal maps of interval), in the terms of Legendre transform of the geometric pressure. This belongs to more general so-called *multifractal spectra* theory. Many questions still wait unanswered.

Task 4. Integral mean spectrum, asymptotic variance.

For every $f : X \to X$ open expanding map and every Hölder $\phi : X \to \mathbb{R}$ the following Ruelle's equality holds, see [Ruelle], [PUbook], [Prz:ICM-18]:

$$\text{(1)} \quad \{\texttt{second\_derivative}\}\ \sigma^2_\mu(\psi) = \frac{d^2P(t\phi)}{dt^2}\bigg|_{t=t_0} \quad \text{for } \sigma^2 = \sigma^2_\mu(\psi) := \lim_{n\to\infty}\frac{1}{n}\int(S_n\psi)^2,$$

where $\psi := \phi - \int\phi\,d\mu$, $S_n\psi := \sum_{j=0}^{n-1}\psi\circ f^j$ and $\mu = \mu_{t_0\phi}$ is the unique equilibrium state for $t_0\phi$.

• OBJECTIVE O4.1. Prove the following (technical) Conjecture: The above formula for σ^2 holds for all rational maps and *hyperbolic potentials* on Julia sets, or just for simply connected RB-domains, $f : \mathrm{Fr}\,\Omega \to \mathrm{Fr}\,\Omega$ and μ equivalent to harmonic measure. Next develop the theory, relating geometric pressure to integral means spectrum for univalent holomorphic functions and asymptotic variance $\sigma^2(\log R')$ in the context of Weil-Petersson metric and the geodesic flow related, on Teichmüller spaces of surfaces (via quasi-Fuchsian groups limit sets) . See [McMullen, Ivrii, Prz:ICM-18]

For a simply connected $\Omega \subset \mathbb{C}$ one considers the *integral means spectrum* depending only on Ω,

$$\text{(2)} \qquad \beta_\Omega(t) := \limsup_{r\to 1}\frac{1}{|\log(1-r)|}\log\int_{\zeta\in\partial\mathbb{D}}|R'(r\zeta)|^t\,|d\zeta|.$$

If Ω is an RB-domain for f and $\phi = -\log|f'|$, if $g(z) = z^d$, with g defined in O4.3 (e.g. Ω being the basin of ∞ for a polynomial f), then it satisfies the equation $\boxed{\beta_\Omega(t) = t - 1 + \frac{P(t)}{\log d}}$, see e.g. [PUbook, Eq. (9.6.2.)] and [Prz:ICM-18].

For $t_0 = 0$ we have $\mu = \widehat{\omega}$ for $\widehat{\omega}$ invariant measure equivalent to harmonic one, and the left hand side of (1) can be written as $(\frac{1}{2}\log d)\sigma^2(\log R')$, where $\sigma^2(\log R') := \limsup_{r\to 1}\frac{\int_{\partial\mathbb{D}}|\log R'(t\zeta)|^2\,|d\zeta|}{-2\pi\log(1-r)|}$. So (1) changes to $\sigma^2(\log R') = 2\frac{d^2\beta_\Omega(t))}{dt^2}|_{t=0}$, [Ivrii], an analytic formula, alike dynamical (1).

• OBJECTIVES O4.2-4. Understand better above $\sigma^2(\log R')$ for various classes of Ω. Continue analysis of *Fine Structure of Harmonic Measure*, following Makarov's fundamental [Makarov1] and [Makarov2] in the multifractal analysis language. Extend this to other measures on $\partial\Omega$ and limit Λ. for geometric coding trees in dynamics setting.

Task 5. Topological finite branched coverings

A theory of such maps $f : V \to U$ for $V \Subset U \subset \overline{\mathbb{C}}$ preserving a compact set $X \subset V$ with shrinking to points pullbacks of sufficiently small discs, called *coarse expanding maps*, has been developed recently by Haissinsky and Pilgrim [HP], M. Bonk, D. Meyer and others, following a theory by W. Thurston. The PI in collaboration with T. Das, G. Tiozzo, M. Urbański, A. Zdunik [DPTUZ] proved recently the existence, uniqueness and probability laws for equilibria of adequate potentials and observables (Hölder continuous with respect to visual metric). This theory does not require a bounded degree for iteration assumption (corresponding to the semi-hyperbolicity in the rational functions setting, [CJY]). This assumption is needed e.g. for the theory of visual metric on adequate hyperbolic graphs Γ for which the domain X is Gromov's boundary: quasi-isometry on Γ vs quasi=symmetry on X, in Mostov's rigidity circumstances. A gct has been used in [DPTUZ] in an ambient space.

• OBJECTIVE O5. a) Link these trees with Γ (find them in Γ). b) Replace semi-hyperbolicity (see Introduction) by weaker Topological Collet-Eckmann, to replace a doubling condition by a *mean doubling condition.* c) Explore the technique of blowing up repelling periodic branching points and their grand orbits to circles, introduced in [DPTUZ], getting Sierpiński-Julia carpets, [HP, Section 4.6]. Contribute to the classification.

Task 6. Real perturbations and chaotic dynamics

• OBJECTIVE O6.1. Consider small *real* smooth perturbations of $z \mapsto z^d$ on a neighbourhood of the unit circle in the plane, see [BSTV]. Prove that the dynamical dimension (supremum of Hausdorff dimensions of invariant measures) on the arising invariant Jordan curve J cannot drop below 1. Prove that except obvious cases $\mathrm{HD}(J) > 1$.

• OBJECTIVE O6.2. Prove that invariant hyperbolic set for a smooth diffeomorphism has Hausdorff dimension equal to the sum of Hausdorff dimensions of stable and unstable slices. In particular for a thin nonlinear transversally non-conformal solenoid in $\mathbb{R}^3$ continue the study started in [HS] and [RS], using an equilibrium measure (SRB on stable foliation), done recently in [MPR]. O6.3 projections, O6.4 thick case and O6.5 higher-dimensional solenoids will be investigated.

2) Significance

Accomplishments in Objective O1.1 and O1.2, O2.1 (periodic orbits) and scenery flow in O2.3 would provide new bridges between holomorphic dynamics and fractal porous wiggly hairy structure of Julia sets and geometric equilibria in the language of geometric analysis and geometric measure theory. Elliptic phenomena may contribute to understanding some celestial mechanics phenomena. O2.3 would complement the knowledge on probability laws in non-uniformly hyperbolic dynamics – interests of leading probabilists. Objective O3.1 on limit sets of geometric coding trees would shed a light to the prime ends of holomorphic univalent maps, in other words boundary behaviour of conformal maps theory, as also O3.2. Objectives O4.1-O4.3 are by definition related to the boundary behaviour so harmonic and complex analysis, making sense even in absence of dynamics. Especially interesting is the use of σ^2 to define Weil-Petersson metric on Teichmüller space of complex surfaces. [McMullen]. In Objective O5 Sierpiński-Julia carpets are investigated from another end than for rational maps; the starting object is a branched covering map and the carpet arises in the complement of periodic singularities and their grand orbits blown up. The objective leads to geometric group theory and understanding a resulting Cayley graph. It leads also to geometric measure theory to understand maximal entropy and other equilibrium measures via the graph. Moreover our investigations would lead to quasiconformal and related classifications depending on recurrence of critical points (semi-hyperbolic or TCE) in geometric analysis. Objective O6.1 on real perturbations of z^2 to solve an old problem whether $\mathrm{HD}(J) > 1$, popularized in particular by the PI, seemed for a long time out of reach in the full generality. Recently however due to a big progress in the study of matrix cocycles and self-affine planar sets [BHR], seems more feasible. Finally Objective O6.2 to prove a conjecture about Hausdorff dimension of hyperbolic sets, seeming feasible due to in particular a progress in [MPR] on solenoids links the project to a mainstream in smooth dynamics and physical Sinai-Ruelle-Bowen measures and regularity (Lipschitz) of (un)stable foliations.
And else, the programme contributes to comprehending the local geometry of fractals around [Man].

3) Work plan

Each of our tasks will be performed during the whole period of the programme, as difficult and related to each other. Actions needed to succeed in each of them have been already roughly presented in the description of Objectives above.

4) Methods

A weakly small seminar on topics of the project will be run at IMPAN by the PI, accompanying a traditional dynamical systems seminar at IMPAN (run now by W. Cordeiro, M. Rams, A. Dudko, Y. Gutman and the PI). Working groups will be organized, also online meetings. Workshops "On geometric complexity of Julia sets-II" later in 2020 and "Hyperbolic interweaving in dynamics" in September 2021, both at IMPAN Będlewo Center, which will stimulate the programme's research are already planned. Cooperation/collaborations is planned with specialists in at least UK (D. Cheraghi, A. DeZotti, S. van Strien), France (F. Bianchi, J. Graczyk, N. Mihalache), Spain (N. Fagella), Israel (G. Levin, O. Ivrii), Finland (A. Käenmäki), USA (S. Rohde, M. Urbański, J. Rivera-Letelier), Canada

(G. Tiozzo), Warsaw (G. Świątek, A. Zdunik, K. Barański, L. Jaksztas and members of the Dynamical Systems Laboratory at IMPAN). The project team will be the PI, two postdocs and a phd student.

References

[BHR] B. Bárány, M. Hochman, A. Rapaport, *Hausdorff dimension of planar self-affine sets and measures*, Inventiones mathematicae 216.3 (2019), 601-659.

[BFU] T. Bedford, A. M. Fisher, M. Urbański, *The scenery flow for hyperbplic Julia sets*, Proc. London Math. Soc. (3), 85.2 (2002), 467-492.

[BMS] I. Binder, N. Makarov, S. Smirnov, *Harmonic measure and polynomial Julia sets*, Duke Math. J. 117.2 (2003), 343 – 365.

[BSTV] B. Bielefield, S. Sutherland, F. Tangerman, J. J. Veerman, *Dynamics of certain nonconformal degree-two maps of the plane* Experimental Math. 3 (1993), 281-300

[BS] D. B. Beliaev, S. K. Smirnov, *On dimension of porous measures*, Math. Ann. 323 (2002), no. 1, 123-141.

[CDY] D. Cheraghi, A. DeZotti, Fei Yang, *Dimension paradox of irrationally indifferent attractors*, arXiv:2003.12340v1.

[CJY] L. Carleson, P. W. Jones, J.-Ch. Yoccoz, *Julia and John*, Bol. Soc. Brasil. Mat. 25.1 (1994), 1-30.

[CorRiv:19] D. Coronel, J. Rivera-Letelier, *Sensitive dependence of geometric Gibbs states at positive temperature.* Comm. Math. Phys. 368.1 (2019), 383 – 425.

[DPTUZ] T. Das, F. Przytycki, G. Tiozzo, M. Urbański, A. Zdunik *Thermodynamic formalism for coarse expanding dynamical systems*, preprint, August 2019, arXiv:1908.08270v1, submitted.

[GJ] J. Graczyk, P. W. Jones. *Dimension of the boundary of quasiconformal Siegel disks*, Inventiones mathematicae 148 (2002), 465-493.

[HP] P. Haissinsky, K. M. Pilgrim, *Coarse Expanding Conformal Dynamics*, Asterisque 325 (2009).

[HS] B. Hasselblatt, J. Schmeling, *Dimension product structure of hyperbolic sets.* In Modern Dynamical Systems and Applications, B. Hasselblatt, M. Brin, Y. Pesin, eds., Cambridge Uni. Press, 2004, pp. 331-345.

[Ivrii] O. Ivrii, *On Makarov's principle in conformal mapping*, Int. Math. Res. Not. 5 (2019), 1543-1567.

[Kaenmaki] A. Käenmäki, *Dynamics of the scenery flow and conical density theorems*, DYnamical Systems (ed. F. Przytycki), Simons Semesters in BC, 2015, Banach Center Publications, vol. 135, Warszawa 2018.

[LPS] G. Levin, F. Przytycki, W. Shen, *The Lyapunov exponent of holomorphic maps*, Inventiones math. 205 (2016), 363 – 382.

[Makarov1] N. G. Makarov *On the distortion of boundary sets under conformal mappings*, Proc. London Math. Soc. 3^d ser. 51 (1985), 369 – 384.

[Makarov2] N. G. Makarov *Fine structure of harmonic measure* Algebra i Analiz 10.2 (1998), 1-62.

[Man] B. Mandelbrot, *The Fractal Geometry of Nature,* W. H. Freeman and Co., San Francisco, Calif., 1982.

[McMullen] C. T. McMullen, *Thermodynamics, dimension and the Weil-Petersson metric*, Invent. Math. 173 (2008), no. 2, 365 – 425.

[MPR] R. Mohammadpour, F. Przytycki, M. Rams, *Hausdorff and packing dimensions and measures for nonlinear transversally non-conformal thin solenoids*, preprint, March 2020, arXiv:2003.08926v1, submitted.

[Prz:94] F. Przytycki, *Accessibility of typical points for invariant measures of positive Lyapunov exponents for iterations of holomorphic maps*, Fundamenta Mathematicae 144 (1994), 259 – 278.

[Prz:06] F. Przytycki, *On the hyperbolic Hausdorff dimension of the boundary of a basin of attraction for a holomorphic map and of quasirepellers*, Bull. Pol. Acad. Sci. Math. 54.1 (2006), 41 – 52.

[Prz:ICM-18] F. Przytycki *Thermodynamic formalism methods in one-dimensional real and complex dynamics* Proceedings of the International Congress of Mathematicians 2018, Rio de Janeiro, Vol.2, pp. 2081-2106

[PrzRiv:11] F. Przytycki, J. Rivera-Letelier, *Nice inducing schemes and the thermodynamics of rational maps*, Communications in Math. Phys. **301.3** (2011), 661 – 707.

[PrzRiv:19] F. Przytycki and J. Rivera-Letelier, *Geometric pressure for multimodal maps of the interval*, Memoirs of the Amer. Math. Soc. 1246 (2019).

[PRS:03] F. Przytycki, J. Rivera-Letelier, S. Smirnov, *Equivalence and topological invariance of conditions for non-uniform hyperbolicity in the iteration of rational maps*, Inventiones math. **151** (2003), 29 – 63.

[PRS:04] F. Przytycki, J. Rivera-Letelier, S. Smirnov, *Equality of pressures for rational functions*, Ergodic Theory Dynam. Systems **24** (2004), 891 – 914.

[PrzRoh] F. Przytycki, S. Rohde, *Porosity of Collet-Eckmann Julia sets*, Fund. Math. 155 (1998), 189-199.

[PSkrzy] F. Przytycki, J. Skrzypczak, *Convergence and pre-images of limit points for coding trees for iterations of holomorphic maps*, Math. Annalen 290 (1991), 425 – 440.

[PUbook] F. Przytycki, M. Urbański, *Conformal Fractals: Ergodic Theory Methods*, London Mathematical Society Lecture Note Series 371, Cambridge University Press, 2010.

[PUZ:89] F. Przytycki, M. Urbański, A. Zdunik, *Harmonic, Gibbs and Hausdorff measures for holomorphic maps, I*, Annals of Math. 130 (1989), 1 – 40.

[PZ:20] F. Przytycki, A. Zdunik, *On Hausdorff dimension of polynomial not totally disconnected Julia sets*, preprint, March 2020, arXiv:2003.12612v2, submitted.

[RS] M. Rams, K. Simon, *Hausdorff and packing measure for solenoids* Erg. Th. Dyn. Sys. 23 (2003), 273-292.

[Ruelle] D. Ruelle, *Thermodynamic Formalism*, Encyclopedia of Mathematics and Its Applications, vol. 5, Addison-Wesley Publ. Co., London, 1978.

[Z1] A. Zdunik, *Parabolic orbifolds and the dimension of the maximal measure for rational maps*, Invent. math. 99 (1990), 627 – 649.

[Z2] A. Zdunik, *Harmonic measure versus Hausdorff measures on repellers for holomorphic maps*, Trans. AMS 326.2 (1991), 633 – 652.