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On measure and Hausdorff dimension of Julia sets for holomorphic Collet-Eckmann maps

Dedicated to the memory of Ricardo Mañé

Introduction.

It is well-known that if $f : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ a rational map of the Riemann sphere is hyperbolic, i.e. expanding on its Julia set $J = J(f)$ namely $|(f^n)'| > 1$ for an integer $n > 0$, then its Hausdorff dimension $\mathrm{HD}(J) < 2$.

The same holds for a more general class of "subexpanding" maps, namely those maps, whose all critical points in $J(f)$ are non-reccurrent, supposed $J(f) \neq \bar{\mathbb{C}}$, see [U] (periodic parabolic points are allowed).

On the other hand there is an abundance of rational maps with $J \neq \bar{\mathbb{C}}$ and $\mathrm{HD}(J) = 2$, [Shi].

Recently Chris Bishop and Peter Jones proved that for every finitely generated not geometrically finite Kleinian groups for the Poincaré limit set Λ one has $\mathrm{HD}(\Lambda) = 2$. As geometrically finite exhibits some analogy to subexpanding in the "Kleinian Groups – Rational Maps" dictionary, the question arised, expressed by Ch. Bishop and M. Lyubich at the MSRI Berkeley conference in January 1995, isn't it true for every non-subexpanding rational map with connected Julia set, that $HD(J) = 2$?

Here we give a negative answer. For a large class of "non-uniformly" hyperbolic so called Collet-Eckmann maps, studied in [P1], satisfying an additional Tsujii condition, $\mathrm{HD}(J) < 2$.

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Notation. For a rational map $f:\bar{\mathbb{C}}\to\bar{\mathbb{C}}$ denote by $\mathrm{Crit}(f)$ the set of all critical points of f, i.e.points where $f'=0$. Let $\nu:=\sup\{$ multiplicity of f^n at $c : c\in \mathrm{Crit}(f)\cap J\}$. Finally denote by $\mathrm{Crit}'(f)$ the set of all critical points of f in $J(f)$ whose forward trajectories do not contain other critical points.

We prove in this paper the following:

Theorem A. Let f be a rational map on the Riemann sphere $f:\bar{\mathbb{C}}\to\bar{\mathbb{C}}$, and suppose there exist $\lambda>1, C>0$ such that for every f-critical point $c\in\mathrm{Crit}'(f)$

$$|(f^n)'(f(c))|\geq C\lambda^n, \tag{0.1}$$

there are no parabolic periodic points, and $J(f)\neq\bar{\mathbb{C}}$.

Then $\mathrm{Vol}(J(f)=0$, where Vol denotes Riemann measure on $\bar{\mathbb{C}}$.

Theorem B. Under the conditions of Theorem A, assume additionally that

$$\lim_{t\to\infty}\limsup_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\max(0,-\log\big(\mathrm{dist}(f^j(c),\mathrm{Crit}(f))\big)-t)=0. \tag{0.2}$$

Then $\mathrm{HD}(J(f))<2$.

For $f(z)=z^2+c,\ c\in[-2,0]$ real, it is proved in [T] that (0.1) and (0.2) are satisfied for a positive measure set of parameters c for which there is no sink in the interval $[c,c^2+c]$. Tsujii's condition in [T], called there *weak regularity*, is in fact apparently stronger than (0.2). The set of subexpanding maps satisfying (0.1) and weak regularity has measure 0, [T]. Thus Theorem B answers Bishop-Lyubich's question.

Remark. In [DPU] it is proved that for every rational map $f:\bar{\mathbb{C}}\to\bar{\mathbb{C}},\ c\in\mathrm{Crit}'$

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}-\log\mathrm{dist}(f^j(c),\mathrm{Crit}(f))\leq C_f$$

where C_f depends only on f. Here in the condition (0.2) it is sufficient, for Theorem B to hold, to have a positive constant instead of 0 on the right hand side, unfortunately apparently much smaller than C_f.

Crucial in proving Theorems A and B is the following

Theorem 0.1 (on the existence of pacim), see [P1]. Let $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ satisfy the assumptions of Theorem A. Let μ be an α-conformal measure on the Julia set $J=J(f)$ for an arbitrary $\alpha>0$. Assume[1]

$$\mu \text{ has no atoms at critical points of } f \tag{0.3}$$

Assume also that there exists $1<\lambda'<\lambda$ such that for every $n\geq 1$ and every $c\in \mathrm{Crit}'(f)$

$$\int \frac{d\mu}{\mathrm{dist}(x,f^n(c))^{(1-1/\nu)\alpha}}<C^{-1}(\lambda')^{\alpha n/\nu}. \tag{0.4}$$

Then there exists an f-invariant probability measure m on J absolutely continuous with respect to μ (pacim).

Recall that a probability measure μ on J is called α-conformal if for every Borel $B\subset J$ on which f is injective $\mu(f(B))=\int_B|f'|^\alpha d\mu$. In particular $|f'|^\alpha$ is Jacobian for f and μ. (A function φ such that $\mu(f(B))=\int_B\varphi d\mu$ for every B as above is called Jacobian.) The number α is called the exponent of the conformal measure.

If $\mathrm{Vol}(J)>0$ then the restriction of Vol to J, normalized, is 2-conformal and obviously satisfies (0.3) and (0.4). If $\mathrm{HD}(J)=2$ then by [P1] we know there exists a 2-conformal measure μ on J but we do not know whether it is not too singular, namely whether it satisfies (0.3) and (0.4). Fortunately for every f satisfying the assumptions of Theorem A and additionally the assumption (0.2) we can prove that (0.4) holds indeed for every α-conformal measure and we can construct a $\mathrm{HD}(J)$-conformal measure satisfying (0.3) repeating the construction from [DU].

Gathering together the results which we prove along the paper and refering to [P1] we obtain the following extension of Theorem B:

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[1] In the Dijon preprint version of [P1] this assumption is missing. I thank J. Graczyk for pointing me out this error.

Theorem C. For every rational map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ satisfying Collet-Eckmann condition (0.1), Tsujii condition (0.2), having no parabolic periodic points and with Julia set J not the whole sphere the following holds: $\mathrm{HD}(J) = \mathrm{Cap}(J) < 2$, there exists a $\mathrm{HD}(J)$-conformal probability measure on J not having atoms at critical points and there exists a probability f-invariant measure m absolutely continuous with respect to μ and such that $dm/d\mu > \mathrm{Const} > 0$. The measure m is ergodic, of positive entropy, and has positive Lyapunov exponent.

(We write $\mathrm{Cap}(J)$ for Minkowski dimension. Other names: box dimension, limit capacity.)

Notation. Const will denote various positive constants which may change from one formula to another, even in one string of estimates.

Section 1. More on pacim. Proof of Theorem A.

Proposition 1.1. In the situation of Theorem 0.1 there exists $K > 0$ such that μ-a.e. $\frac{dm}{d\mu} \geq K$.

Proof. In Proof of Theorem 0.1 [P1] one obtains m as a weak* limit of a subsequence of the sequence of measures $\frac{1}{n}\sum_{j=0}^{n-1} f_*^j(\mu)$.

It is sufficient to prove that there exists $K > 0$ and $n_0 > 0$ such that for μ-a.e. $y \in J(f)$

$$\frac{df_*^n(\mu)}{d\mu}(y) = \mathcal{L}^n(1\!\!1) \geq K. \tag{1.1}$$

Here $\mathcal{L}$ denotes the transfer operator, which can be defined for example by $\mathcal{L}(\varphi)(y) = \sum_{f(z)=y} |f'(z)|^{-\alpha}\varphi(z)$. $1\!\!1$ is the constant function of value 1. We can assume $y \notin \bigcup_{n>0} f^n(\mathrm{Crit}(f))$ because

$$\mu(\bigcup_{n>0} f^n(\mathrm{Crit}(f))) = 0 \tag{1.2}$$

If a critical value for f^n were an atom then a critical point would have μ measure equal to ∞.

(The equality in (1.1) follows from the definition of $\mathcal{L}$. However pay attention that it makes use of the assumption (0.4) if one considers $\mathcal{L}\mathbb{1}$ as a classical function.)

It is sufficient to prove the inequality (1.1) for $y \in B(x,\delta)\cap J(f)$ for an *a priori* chosen x and an arbitrarily small δ and next to use the fact that there exists $m \geq 0$ such that $f^m(B(x,\delta)) \supset J(f)$ (called *topological exactness*). Indeed

$$\mathcal{L}^n(\mathbb{1})(w) = \sum_{f^m(y)=w} \mathcal{L}^{n-m}(\mathbb{1})(y)|(f^m)'|^{-\alpha} \geq \ (\sup|(f^m)'|)^{-\alpha}\mathcal{L}^{n-m}(y_0)$$

where $y_0 \in f^{-m}(\{w\}) \cap B(x,\delta)$.

Recall the estimate from [P1]. For an arbitrary $\gamma > 1$ there exists $C > 0$ such that for every $x \in J(f)$

$$\mathcal{L}^n(\mathbb{1})(x) \leq C + C \sum_{c\in \mathrm{Crit}(f)\cap J} \sum_{j=0}^{\infty} \frac{\gamma^j \lambda^{-j\alpha/\nu}}{\mathrm{dist}(x, f^j(f(c)))^{(1-1/\nu)\alpha}}. \tag{1.3}$$

By the assumptions (0.1) and (0.3) the above function is μ-integrable if γ is small enough.

Pay attention to the assumption (0.3). It concerns only $c \in \mathrm{Crit}'$. Fortunately there is only a finite number of summands in (1.3) for which $f^{j_0}(c) \in \mathrm{Crit}, j_0 \geq j$. Each summand is integrable because up to a constant it is bounded by $\mathcal{L}^j(\mathbb{1})$.

So

$$\sum_{c\in \mathrm{Crit}(f)\cap J} \sum_{j=s}^{\infty} \frac{\gamma^j \lambda^{-j\alpha/\nu}}{\mathrm{dist}(x, f^j(f(c)))^{(1-1/\nu)\alpha}} \to 0 \quad \mu - \text{a.e. as } s \to \infty. \tag{1.4}$$

Fix from now on an arbitrary $x \in J(f)$ for which (1.4) holds, $(dm/d\mu)(x) \geq 1$ and $x \notin \bigcup_{n>0} \varphi^n(\mathrm{Crit}(f))$ (possible by (1.2) and by $\int (dm/d\mu)d\mu = 1$).

We need now to repeat from [P1] a part of the Proof of Theorem 0.1:

For every $y \in B(x,\delta)$ and $n > 0$

$$\mathcal{L}^n(\mathbb{1})(y) = \sum_{y'\in f^{-n}(y),\text{regular}} |(f^n)'(y')|^{-\alpha} + \sum_{(y',s)\text{singular}} \mathcal{L}^{n-s}(\mathbb{1})(y')|(f^s)'(y')|^{-\alpha}$$

$$= \sum_{\text{reg},y} + \sum_{\text{sing},y} . \tag{1.5}$$

We shall recall the definitions of *regular* and *singular*: Take an arbitrary subexponentially decreasing sequence of positive numbers $b_j, j = 1, ...$ with $\sum b_j = 1/100$. Denote by $B_{[k}$ the disc $B(x, (\prod_{j=1}^{k}(1-b_j))2\delta)$. We call s the *essentially critical time* for a sequence of compatible components $W_j = \mathrm{Comp} f^{-j}(B_{[j})$, where compatible means $f(W_j) \subset W_{j-1}$, if there exists a critical point $c \in W_s$ such that $f^s(c) \in B_{[s}$.

We call y' *regular* in (1.5) if for the sequence of compatible components $W_s, s = 0, 1, ..., n, W_n \ni y'$ no $s < n$ is essentially critical.

We call a pair (y', s) *singular* if $f^s(y') = y$ and for the sequence of compatible components $W_j, j = 0, 1, ..., s, W_s \ni y'$ the integer s is the first (i.e. the only) essentially critical time.

If δ is small enough then all s in $\sum_{\mathrm{sing},x}$ are sufficiently large that $\sum_{\mathrm{sing},x} \leq 1/2$. This follows from the estimates in [P1, Sec.4]; here is the idea of the proof: Transforming $\sum_{\mathrm{sing},x}$ in (1.5) using the induction hypothesis (1.3) we obtain the summands

$$C \frac{\gamma^j \lambda^{-j\alpha/\nu}}{\mathrm{dist}(x, f^{s+j-1}(f(c)))^{(1-1/\nu)\alpha}}, \quad j = 0, ..., n-s$$

multiplied by

$$\mathrm{Const}|(f^{s-1})'(x')|^{-\alpha/\nu} a_s < \gamma^{s-1}\lambda^{-(s-1)\alpha/\nu}.$$

The numbers a_s are constants arising from distortion estimates, related to b_s. The numbers γ^s swallow them and other constants.

(There is a minor inaccuracy here: (s, x') is a singular pair where the summand appears, provided the captured critical point c is not in the forward trajectory of another critical point, otherwise one moves back to it, see [P1] for details.)

Now $\sum_{\mathrm{sing},x} \leq 1/2$ follows from (1.4).

The result is that $\sum_{\mathrm{reg},x} \geq 1/2$. So by the uniformly bounded distortion along regular branches of f^{-n} on $B(x, \delta)$ we obtain

$$\mathcal{L}^n(\mathbb{1})(y) \geq \sum_{\mathrm{reg},y} \geq \mathrm{Const} \sum_{\mathrm{reg},x} \geq \mathrm{Const} > 0$$

The name *regular* concerned formally $y' \in f^{-n}(y)$ but in fact it concerns the branch of f^{-n} mapping y to y' not depending on $y \in B(x, \delta)$.

By distortion of any branch g of f^{-n} on a set U we mean $\sup_{z\in B}|g'(z)|/\inf_{z\in B}|g'(z)|$.
Proposition 1.1 has been proved. ♣

There exists a decomposition of J in a finite number of ergodic components $E_1, ..., E_k$, see [P1, Theorem B]. Denote the measure m restricted to E_i and normalized, by m_i. Taking this into account we obtain

Corollary 1.2 In the situation of Theorem 0.1 for measure-theoretic entropy $\mathrm{h}_{m_i}(f) > 0$, for every $i = 1, ..., k$.

Proof. Denote $dm/d\mu$ by u.

Consider an open set $U \subset \overline{\mathbb{C}}$ intersecting $J(f)$ such that there exist two branches g_1 and g_2 of f^{-1} on it. Then by the f-invariance of m we have $\mathrm{Jac}_m(g_1) + \mathrm{Jac}_m(g_2) \leq 1$ ($= 1$ if we considered all branches of f^{-1}). $\mathrm{Jac}_m(g_i)$ means Jacobian with respect to m for g_i.

We have $m(U) > 0$ because μ does not vanish on open sets in J (by the topological exactness of f on J) and by Proposition 1.2. At m-a.e. $x \in U$

$$\mathrm{Jac}_m(g_i)(x) = u(g_i(x))|g_i'(x)|u(x)^{-1} > 0,$$

(here we also used (1.4)).

Hence $\mathrm{Jac}_m(g_i) < 1$, so $\mathrm{Jac}_m(f) > 1$ on the set $g_i(U)$, $i = 1, 2$. Now we use Rochlin's formula and obtain

$$\mathrm{h}_{m_i}(f) = \int_{E_i} \log \mathrm{Jac}_m(f) dm_i > 0$$

♣

Let $\chi_{m_i} = \int \log |f'| dm_i$ denote the Lyapunov characteristic exponent on E_i.
Corollary 1.3 In the situation of Theorem 0.1, $\chi_{m_i} > 0$ for every $i = 1, ..., k$.

Proof. This Corollary follows from Ruelle's inequality $\mathrm{h}_{m_i}(f) < 2\chi_{m_i}$, see [R].

Proof of Theorem A. Suppose $\mathrm{Vol}(J(f)) > 0$. After normalization we obtain a 2-conformal measure μ on $J(f)$ and by Theorem 0.1 and Corollary 1.3 a pacim m with $\chi_m > 0$. By Pesin's Theory [Pesin] in the iteration in the dimension 1 case [Le] ([Le] is on the real case, but the complex one is similar), for m-a.e. x, there exists a sequence of integers $n_j \to \infty$ and $r > 0$ such that for every j there exists a univalent branch g_j of f^{-n_j} on $B_j := B(f^{n_j}(x), r)$ mapping $f^{n_j}(x)$ to x and g_j has distortion bounded by a uniform constant. By $\chi_m > 0$ $\mathrm{diam} g_j(B(f^{n_j}(x), r) \to 0$. (This follows also automatically from the previous assertions by the definition of Julia set [GPS].) Now we can forget about the invariant measure m and go back to Vol. Because $J(f)$ is nowhere dense in $\overline{\mathbb{C}}$, there exists $\varepsilon > 0$ such that for every $z \in J(f)$

$$\frac{\mathrm{Vol}(B(z,r) \setminus J(f))}{\mathrm{Vol}(B(z,r))} > \varepsilon.$$

Bounded distortion for g_j on $B(z,r)$, $z = f^{n_j}(x)$ allows to deduce that the same part of each small disc$\approx g_j(B_j)$ around x is outside $J(f)$, up to multiplication by a constant. This is so because we can write for every $X \subset B(z,r), y \in B(z,r)$

$$\mathrm{Vol}(g_j(X)) \approx |g_j'(y)|^2 \mathrm{Vol}(X) \tag{1.6}$$

where $\approx$ means up to the multiplication by a uniformly bounded factor. So x is not a density point of $J(f)$. On the other hand a.e. point is a density point. So $\mathrm{Vol} J(f) = 0$ and we arrived at a contradiction. ♣

Section 2. Proof of Theorem B.

Definition. We call a probability measure μ on J α-subconformal if the equality in the definition of α-conformal measure (see Introduction) is replaced by the inequality: $\mu(f(B)) \geq \int_B |f'|^\alpha d\mu$.

Lemma 2.1. Suppose f satisfies the assumptions of Theorem B. Then for every $\beta, \sigma > 0$ there exists $C_1 > 0$ such that for every $c \in \mathrm{Crit}'$ and $n_0 > 0$ there exists a sequence $r_j, j = 1, 2, ...$ satisfying

$$r_1 > C_1 \exp -\beta n_0 \tag{2.1}$$

$$r_{j+1} > r_j^{1+\sigma} \tag{2.2}$$

$$r_{j+1} < r_j/2. \tag{2.3}$$

and

$$\mu(B(f^{n_0}(c), r_j)) \leq C_1 r_j^\alpha \tag{2.4}$$

for every $\alpha \leq 2$ and every α-subconformal measure μ.

Proof. Step 1. Denote the expression from (0.2)

$$\max\big(0, -\log \inf_{c\in \mathrm{Crit}'(f)} \mathrm{dist}(f^n(c), \mathrm{Crit}(f)) - t\big)$$

by $\varphi_t(n)$. Consider the following union of open-closed intervals

$$A'_t := \bigcup_n (n, n + \varphi_t(n) \cdot K_f] \quad \text{and write} \quad A_t := \mathbb{Z}_+ \setminus A'_t \ ,$$

for an arbitrary constant $K_f > \nu / \log \lambda$ (in the convention that if $\varphi_t(n) = 0$, then the interval in the union is empty).

By (0.2) for every $a > 0$ there exist $t > 0$ and $n(a,t)$ such that for every $n \geq n(a,t)$

$$A_t \cap [n, n(1+a)] \neq \emptyset \tag{2.5}$$

Moreover, fixing an arbitrary integer $M > 0$, we can guarantee for every $n' \geq n(1+a)$, $n \geq n(a,t)$

$$\sharp(A_t \cap \{j \in [n, n'] : j \text{ divisible by } M\}) \geq \frac{1}{2M}(n' - n). \tag{2.6}$$

Observe that for every a, n_0, n

$$[n_0 + n, n_0 + n + a(n_0 + n)] = [n_0 + n, n_0 + n + a(\frac{n_0}{n} + 1)n].$$

So if $n \geq bn_0$ for an arbitrary $b > 0$ and $n_0 \geq n(a,t)$, then (2.5) yields

$$A_t \cap [n_0 + n, n_0 + n + a(b^{-1} + 1)n] \neq \emptyset. \tag{2.7}$$

Denote in the sequel $a(b^{-1} + 1)$ by a'.

Step 2. Observe now that if $n \in A_t$ then for every $c \in \mathrm{Crit}'(f)$ there exist branches $g_s, s = 1, 2, ...n-1$ of f^{-s} on $B_n := B(f^n(c), \delta)$ such that $g_s(f^n(c)) = f^{n-s}(c)$, distortions bounded by a uniform constant C_2 (i.e. $\sup |g_s'| / \inf |g_s'| \leq C_2$), where $\delta = \varepsilon \exp -t\nu$, for a constant ε small enough. Sometimes to exhibit the dependence on n we shall write $g_{s,n}$.

Indeed, define g_s on $B_{[s} = B(f^n(c), \prod_{j=1}^{s}(1-b_j)2\delta)$ for $s = 1, 2, ...n-1$ according to the procedure described in the Proof of Proposition 1.1. If there is an obstruction, namely s an essential critical time, then for every $z \in B_{[s}$

$$|g_{s-1}'(z)| \leq \lambda^{-s}\vartheta^s \leq \exp(-s\nu/K_f) \tag{2.8}$$

for $\vartheta > 1$ arbitrarily close to 1 (in particular such that $K_f > \frac{\nu}{\log\lambda - \log\vartheta}$) and for s large enough. The constant ϑ takes care of distortion. (2.8) holds for $z = f^s(q)$, where q is the critical point making s a critical time, without ϑ by (0.1) (with the constant C instead). The small number ε takes care of s small, which cannot then be essential critical.

The inequality (2.4) and rooting ($1/\nu$ to pass from $s-1$ to s) imply $\varphi_t(f^{n-s}(c)) \geq s/K_f$, so $n \notin A_t$, a contradiction.

Step 3. We find r_j satisfying the assertions of the Lemma by taking

$$r_j := \frac{1}{2C_2} \mathrm{diam}\; g_{n_j, n_0+n_j}(B(f^{n_0+n_j}(c), \delta))$$

where n_j are taken consecutively so that $n_0 + n_j \in A_t$ and

$$n_{j+1} \in [(1+\vartheta)n_j, (1+\vartheta)n_j(1+a')] \quad \text{for} \quad j \geq 2 \quad \text{and}$$

$$n_1 \in [bn_0, bn_0 + a'bn_0],$$

where $\vartheta > 0$ is an arbitrary constant close to 0. This is possible by (2.7).

This gives for say $\vartheta < a' < 1$

$$r_{j+1}/r_j \geq C_2^{-1} \exp(-3(\log L)a'n_j), \tag{2.9}$$

where $L := \sup |f'|$. One obtains this in 2 steps: first by the branch $g_{n_{j+1}-n_j, n_0+n_{j+1}}$, next by g_{n_j, n_0+n_j} which shrinks the ratio by at most C_2^{-1}. In the same way by acting only by g_{n_1, n_0+n_1} one obtains (2.1).

To conclude we need to know that r_j shrink exponentially fast with $n_j \to \infty$, uniformly on n_0. For that we need the following fact (see for example [GPS], find the analogous fact in the Proof of Theorem A):

(*) For every $r > 0$ small enough and $\xi, C > 0$ there exists m_0 such that for every $m \geq m_0, x \in J(f)$ and a branch g of f^{-m} on $B(x,r)$ having distortion less than C, we have diam $g(B(x,r)) < \xi r$.

Apply now (2.6) to $n = n_0, n' = n_j + n_0$. We obtain a "telescope": For all consecutive $\tau_1, \tau_2, ...\tau_{k(j)} \in A_t \cap [n_0, n_j + n_0]$ divisible by M

$$g_{\tau_{i+1}-\tau_i,\tau_{i+1}}(B(f^{\tau_{i+1}}(c),\delta)) \subset B(f^{\tau_i}(c),\delta/2C_2)$$

for $M \geq m_0$ from (*).

Hence using (2.6)

$$r_j \leq 2^{-n_j/2M} \ . \tag{2.10}$$

The property (2.3) follows from the fact that for n_0 large enough, for every $j > 0$, $n_{j+1} - n_j \geq M$ and the argument the same as for $\tau_{i+1} - \tau_i$ above is valid.

Denote $2a' \log L$ by γ and $(\log 2)/2M$ by γ'. (2.9) and (2.10) give

$$r_{j+1} \geq C_2^{-1} r_j \exp -\gamma n_j \geq C_2^{-1} r_j (\exp -\gamma' n_j)^{\gamma/\gamma'} \geq C_2^{-1} r_j^{1+\gamma/\gamma'}.$$

As γ' is a constant and γ can be made arbitrarily small if a is small enough, we obtain (2.2). C_2^{-1} disappears when we double γ/γ' for δ small enough.

Finally we obtain $\mu(B(f^{n_0}(c),r_j)) \leq \mu g_{n_j,n_0+n_j}(B(f^{n_0+n_j}(c),\delta)) \leq C_2^{\alpha}\delta^{-\alpha}r_j^{\alpha}$ what proves (2.4).

We have proved the Lemma for every n_0 large enough. Now by pulling back one easily provees it for every $n_0 > 0$. ♣

Remark 2.2. The only result at our disposal on the abundance of non-subexpanding maps satisfying (0.1) and (0.2) is Tsujii's one concerning $z^2 + c$, c real (see the Introduction). For this class however the exponential convergence of diamComp$f^{-n_j}(B(f^{n_j+n_0}(0),\delta)$ to 0 follows from [N] (the component containing $f^{n_0}(c)$). So restricting our interests to this class we could skip (2.6) and the considerations leading to (2.10) above.

By [N] $\operatorname{diam}\big(\operatorname{Comp}(f^{-n}(B(x,\delta))) \cap I\!\!R\big) < C\tilde{\lambda}^{-n}$ for some constants $C>0, \tilde{\lambda} > 1, \delta$ small enough and every component Comp. Just the uniform convergence of the diameters to 0 as $n \to \infty$ follows from [P1], but I do not know how fast it is.

Lemma 2.3 Under the assumptions of Theorem B, for every $\lambda' > 1$ there exists $C > 0$ such that for every $\alpha \leq 2$ and α-conformal measure μ the estimate (0.4) holds.

Proof. By Lemma 2.1 we obtain

$$\int \frac{d\mu}{\operatorname{dist}(x, f^{n_0}(c))^{(1-1/\nu)\alpha}}$$

$$\leq \mu(\overline{\mathbb{C}} \setminus B(f^{n_0}(c), r_1)) \frac{1}{r_1^{(1-1/\nu)\alpha}} + \sum_{j\geq 2} \mu(B(f^{n_0}(c), r_{j-1}) \setminus B(f^{n_0}(c), r_j)) \frac{1}{r_j^{(1-1/\nu)\alpha}}$$

$$\leq \operatorname{Const} \exp(\beta n_0 (1-1/\nu)\alpha) + \operatorname{Const} \sum_{j\geq 2} \frac{r_{j-1}^{\alpha}}{r_j^{(1-1/\nu)\alpha}}$$

$$\leq (\exp(\beta(1-1/\nu)\alpha))^{n_0} + \operatorname{Const} \sum_{j\geq 2} r_{j-1}^{\alpha} r_{j-1}^{-(1-1/\nu)\alpha(1+\sigma)}.$$

The latter series has summands decreasing exponentially fast for σ small enough so it sums up to a constant, hence the first summand dominates. We obtain the bound by $(\lambda')^{n_0}$ with $\lambda' > 1$ arbitrarily close to 1. Thus (0.4) has been proved. ♣

For an arbitrary rational map f restricted to a forward invariant set $K \subset J$ we write $\mathrm{HD}_{\mathrm{ess}}(K)$ for the essential Hausdorff dimension, which can be defined for example as the supremum of the Hausdorff dimension of all expanding isolated Cantor sets in K. (We say that an f-invariant set X is isolated if every forward f-trajectory which starts in a sufficiently small neighbourhood U of X either is contained in X or escapes from U.) There always exists an α-conformal measure with the exponent $\alpha = \mathrm{HD}_{\mathrm{ess}}(J)$, this is the minimal possible exponent for conformal measures, see [DU] [P2] and [PUbook]. If f satisfies (0.1) then $\mathrm{HD}_{\mathrm{ess}}(J) = \mathrm{HD}(J)$, see [P1].

In our situation we can say more:

Lemma 2.4. If f satisfies the assumptions of Theorem B, then there exists an α-conformal measure with $\alpha = \mathrm{HD}_{\mathrm{ess}}(J) = \mathrm{HD}(J)$ which does not have atoms at f-critical points.

Proof. We repeat the construction from [DU]. Consider for every $n = 1, 2, ...$ the set $V_n = B(\mathrm{Crit}', \frac{1}{n})$ and construct μ_n a subconformal measure on $K(V_n) = J \setminus \bigcup_{k\geq 0} f^{-k}(V_n)$ as in [DU, Lemma 5.1].

Here the situation is easier than in [DU] because f on $K(V_n)$ is expanding, [P1, Sec.3]. So each μ_n is α_n-subconformal (α_n-conformal on sets disjoint with $\mathrm{cl}V_n$), with $\alpha_n = \mathrm{HD}_{\mathrm{ess}}(K(V_n))$, $\alpha_n \nearrow \alpha$ and $\mu_n \to \mu$ which is an α-conformal measure.

(In [DU] one obtains each μ_n with $\frac{df_*\mu_n}{d\mu_n} \geq e^{c_n}|f'|^{\alpha_n}$ with $c_n \searrow 0$. Here $c_n = 0$. Also μ in [DU] can have an atom at a critical value. Here, due to (0.1) and the subconformality, this is automatically excluded, otherwise the measure of the forward trajectory of the critical value would be infinite.)

By Lemma 2.1 for every $c \in \mathrm{Crit}'$ we have $\mu_n(B(f(c), r_j) \leq C_1 r_j^{\alpha_n}))$. So

$$\mu_n(\mathrm{Comp} f^{-1}(B(f(c), r_j) \setminus B(f(c), r_{j+3})$$

$$\leq \mathrm{Const}\ r_{j+3}^{(1/\nu(c)-1)\alpha_n} \mu_n(B(f(c), r_j) \setminus B(f(c), r_{j+3}))$$
$$\leq \mathrm{Const} C_1 r_{j+3}^{(1/\nu(c)-1)\alpha_n} r_j^{\alpha_n} = \mathrm{Const} C_1 r_{j+3}^{(1/\nu(c)-3\sigma)\alpha_n}.$$

again using Lemma 2.1. $\nu(c)$ is the multiplicity of f at the critical point c. Comp means the component close to c. $\sigma \approx 0$. It is crucial that the estimate is uniform on n.

Thus one obtains

$$\mu(\mathrm{Comp} f^{-1}(B(f(c), r_{j+1}) \setminus B(f(c), r_{j+2}))) \leq \mathrm{Const}\ r_{j+3}^{(1/\nu(c)-3\sigma)\alpha} \to 0 \qquad (2.11)$$

as $j \to \infty$. (We passed from $j, j+3$ to $j+1, j+2$ to cope with the case $\mu\partial(\mathrm{Comp} f^{-1}(B(f(c), r_{j+1}) \setminus B(f(c), r_{j+2}))) > 0$. Remember that to conclude $\lim \mu_n B = \mu B$ one assumes $\mu(\partial B) = 0$.)

Similarly by further pulling back one obtains (2.11) around critical points in $J \setminus \mathrm{Crit}'$. Finally by the construction μ_n have no atoms at critical points, because the topological supports of μ_n's do not contain critical points. The Lemma has been proved. ♣

Proof of Theorems B and C. By Lemma 2.4 there exist a $\mathrm{HD}(J)$-conformal measure μ on J satisfying (0.3). By Lemma 2.3 μ satisfies also (0.4). Hence by Theorem 0.1 there exists a pacim $m \ll \mu$. Moreover $\chi_m > 0$ by Corollary 1.3. As in the Proof of Theorem A, by Pesin Theory there exists $X \subset J$, $m(X) = \mu(X) = 1$, such that for every $x \in X$ there exists a sequence of integers $n_j(x) \to \infty$, $r > 0$ and univalent branches g_j of f^{-n_j} on $B(f^{n_j}(x), r)$ mapping f^{n_j} to x with uniformly bounded distortion. Write $B_{x,j} := g_j(B(f^{n_j}(x), r))$.

Suppose now that $\mathrm{HD}(J) = 2$. We obtain for every $x \in X$ by applying (1.6) to Vol and μ (similarly as in the Proof of Theorem A)

$$\mu(B_{x,j}) \leq \mathrm{Const}\mathrm{Vol}(B_{x,j}) \leq \mathrm{Const}\ \mathrm{Vol}(B(x, \mathrm{diam} B_{x,j})).$$

If $\mathrm{Vol}X = 0$ then there exists a covering of X by discs $B(x_t, \mathrm{diam} B_{x_t,j_t}), t = 1, 2, ...$ whose union has Vol $< \varepsilon$ for ε arbitrarily close to 0, of multiplicity less than a universal constant (Besicovitch's theorem). Hence

$$\varepsilon \geq \mathrm{Const} \sum_t \mathrm{Vol} B(x_t, \mathrm{diam} B_{x_t,j_t}) \geq \mathrm{Const}\mu(\sum_t B_{x_t,j_t}) \geq 1,$$

a contradiction. Hence $\mathrm{Vol}J \geq \mathrm{Vol}X > 0$.

This contradicts Theorem A that $\mathrm{Vol}J = 0$ and the proof of Theorem B is over.

Remark that we could end the proof directly: As in the Proof of Theorem A we show that no point of X is a point of density of the Vol measure. Hence $\mathrm{Vol}X = 0$. (I owe this remark to M. Urbański.)

To finish the proof of Theorem C it remains only to check the ergodicity. However the ergodicity follows easily from [P1, Sec.3], passing (acting by iterates of f) from a neighbourhood of a.e. point x to a neighbourhood of a critical point, and from the Proof of Lemma 2.1. Briefly: the existence of the branches $g_{n-1,n}$ for a growing sequence of n's yields for every invariant set A with $m(A) > 0$ and every $c \in$ Crit , the existence of $r_n \to 0$ such that $\frac{m(B(c,r_n)\cap A)}{m(B(c,r_n))} \geq \mathrm{Const} > 0$. So x cannot be a point of density of $J \setminus A$. If $m(J \setminus A) > 0$ then similarly x cannot be a point of density of A. This can happen only for a set of x's of measure 0. A contradiction. Theorem C is has been proved. ♣

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