On the transfer operator for rational functions on the Riemann sphere

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Abstract: Let T be a rational function of degree ≥ 2 on the Riemann sphere. Denote $\mathcal{L}_\phi$ the transfer operator of a Hölder-continuous function ϕ on its Julia set $J = J(T)$ satisfying $P(T,\phi) > \sup_{z \in J} \phi(z)$. We study the behavior of $\{\mathcal{L}_\phi^n \psi : n \geq 1\}$ for Hölder- continuous functions ψ and show that the sequence is (uniformly) norm-bounded in the space of Hölder-continuous functions for sufficiently small exponent. As a consequence we obtain that the density of the equilibrium measure μ for ϕ with respect to the $\exp[P(T,\phi) - \phi]$-conformal measure is Hölder-continuous. We also prove that the rate of convergence of $\mathcal{L}_\phi^n \psi$ to this density in sup-norm is $\mathrm{O}\big(\exp(-\theta\sqrt{n})\big)$. ¿From this we deduce the central limit theorem for ψ.

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§1. Introduction

The existence of equilibrium measures for analytic endomorphisms T of the Riemann sphere $\overline{\mathbb{C}}$ has been established in [3]. It has been shown that the transfer operator (Perron-Frobenius-Ruelle operator) acting on the space of continuous functions on the Julia set is almost periodic. This establishes the continuity of the density function with respect to some canonical reference measure (the associated conformal measure). Later, a different argument for this was given in [8], and it has been shown there that the density has at least a logarithmic modulus of continuity (in fact is already Hölder-continuous in some important cases of rational maps with critical points in their Julia set). In this note we continue the investigation of the transfer operator and study the long time behavior of T in more detail.

To begin with let us recall some facts about the thermodynamic formalism for rational functions. The pressure of a continuous function f on the Julia set $J = J(T)$ is denoted by $P(T, f)$ (see [2] for a definition and properties). An equivalent definition is given by the variational principle

$$P(T, f) = \sup\{h_\nu(T) + \int f \, d\nu : \nu \circ T^{-1} = \nu\}.$$

Let $\phi : J \to \mathbb{R}$ be a Hölder continuous function satisfying

$$P(T, \phi) > \sup_{z \in J} \phi(z).$$

In this situation there exists a normalized measure m on J so that the (local) Jacobian of $m \circ T$ with respect to m is given by

$$\frac{dm \circ T}{dm}(x) = \exp[P(T, \phi) - \phi(x)] \quad \text{for } m\text{-a.e. } x.$$

This measure is called $\exp[P(T, \phi) - \phi]$-conformal, or conformal for short in this note. It is the unique $\exp[c - \phi]$-conformal measure, where c may be any constant. It is easy to see that m is a fixed point for the dual of the transfer operator $\mathcal{L}_\phi : C(J(T)) \to C(J(T))$ defined by

$$\mathcal{L}_\phi f(x) = \sum_{T(y)=x} f(y) \exp[\phi(y) - P(T, \phi)] \qquad f \in C(J), \quad x \in J.$$

(Here preimages of critical values are counted with their multiplicities.) It is known that $\mathcal{L}_\phi$ is almost periodic on $C(J(T))$ and the space of eigenfunctions (for eigenvalues of modulus 1) is one-dimensional. It is easy to verify that 1 is the only eigenvalue of modulus 1 and that the corresponding eigenfunction is the density of an invariant measure. This measure is unique up to multiplication and its normalized version is denoted by μ. One can also show that μ is the unique measure maximizing the pressure:

$$P(T,\phi) = h_\mu(T) + \int \phi \, d\mu.$$

Denote

$$h = \frac{d\mu}{dm}$$

the density.

The modulus of continuity

$$w(\epsilon) := \sup_{x,y \in J(T)} \{h(x) - h(y) : \ \mathrm{dist}(x,y) \le \epsilon\}$$

of h has been estimated in [8]: For N large enough there exists a constant $C(N)$ such that

$$w(\epsilon) \le C(N) \left(-\log \epsilon\right)^{-N}. \tag{1.1}$$

Moreover (see [8]), h is Hölder-continuous if the ω-limit set of critical points does not contain critical points from the Julia set.

For $\tau > 0$ denote by $\mathcal{H}_\tau$ the space of Hölder-continuous functions on $J(T)$ equipped with the norm

$$\|f\|_\tau = \|f\|_\infty + \sup_{x \ne y \in J} \frac{|f(x) - f(y)|}{|x-y|^\tau} \qquad (f \in \mathcal{H}_\tau),$$

where $\|f\|_\infty$ denotes the sup-norm in $C(J(T))$. Based on a detailed analysis of return times to annuli with center a critical point (Lemma 2.3) and an estimate of the growth of the diameters of components of preimages of a small disc (Lemma 3.3), we are able to show that for every $f \in \mathcal{H}_\alpha$ all $\mathcal{L}_\phi^n(f)$ belong to $\mathcal{H}_\tau$ and are uniformly bounded:

$$\sup_{n\geq 1} \|\mathcal{L}_\phi^n f\|_\tau < \infty \tag{1.2}$$

for sufficiently small $\tau > 0$.

It has been proved in [3] that $\mathcal{L}_\phi$ is almost periodic on $C(J(T))$, i.e. each family $\mathcal{L}_\phi^n f$ is equicontinuous, and in [8] that (1.1) is a bound for the common modulus of continuity. So (1.2) improves both results. In particular it follows that $h \in \mathcal{H}_\tau$.

We remark that one can show $\mathcal{L}_\phi^n(f) \in \mathcal{H}_\tau$ for $\tau \leq \frac{1}{\nu}\min(\tau(\phi), \alpha)$, where $\tau(\phi)$ is the Hölder exponent of ϕ and ν is the maximal multiplicity of iterates of T at critical points belonging to J. To obtain the boundedness (1.2) in general, τ needs to be taken smaller.

For hyperbolic or parabolic rational maps one may take $\tau = \min(\tau(\phi), \alpha)$, (cf. [1] or [10] for the hyperbolic case which is similar to the case of a one-sided topological Markov chain, and [4] for the parabolic case).

¿From (1.2) we obtain the rate of convergence in sup-norm. We show in section 4 that for Hölder-continuous functions f there exist constants $C > 0$ and $\theta > 0$ such that

$$\|\mathcal{L}_\phi^n f - h \int f \, d\mu\|_\infty \leq C \exp[-\theta\sqrt{n}] \qquad (n \geq 1).$$

Finally, in section 5, we derive the central limit theorem for Hölder-continuous functions f using Gordin's martingale approximation method ([5]): There exists a constant $\sigma^2 \geq 0$, such that

$$\sup_{t\in\mathbb{R}} \left| \mu\left(\left\{z : \frac{1}{\sigma\sqrt{n}} \sum_{i=0}^{n-1} \left(f(T^i(z)) - \int f \, d\mu\right) \leq t\right\}\right) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp[-\frac{u^2}{2}] \, du \right| \to 0,$$

whenever $\sigma^2 > 0$. This result is known in the case of a hyperbolic rational map ([1], it reduces to the case of a one-sided topological Markov chain using a Markov partition) and also in the case of a parabolic rational map ([4]). We also remark that theorems of this type have been proven for maps of the interval (see for example [6] and [11]). In this context Gordin's method has been used also in [7].

§2. Local behavior near critical points

Let $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map of degree ≥ 2. Derivatives and distances are considered in the standard spherical Riemann metric on the Riemann sphere $\overline{\mathbb{C}}$.

Denote the set of critical points for T by $\mathrm{Crit}(T) =: \{z \in \overline{\mathbb{C}} : T'(z) = 0\}$. $B(x, r)$ denotes the open ball of radius r centered at x. For every $c \in \overline{\mathbb{C}}$ define a function $k_c : \overline{\mathbb{C}} \to \{0, 1, 2, ...\} \cup \{\infty\}$ setting

$$k_c(x) = \min\{n \geq 0 : x \notin B(c,\ ae^{-(n+1)})\}.$$

and $k_c(x) = \infty$ if $x = c$. Here a is an arbitrary positive number such that $a < \mathrm{diam}\,\overline{\mathbb{C}}$.

The following lemma can be easily deduced from the fact that up to a biholomorphic change of coordinates every holomorphic function is of the form $z \mapsto z^q$ in some neighborhood of a critical point of order $q \geq 2$.

Lemma 2.1. There exist $\theta > 0$ (depending on a) and $\alpha > 0$ such that

$$e^{-\theta} e^{-\alpha \max\{k_{\tilde{c}(x)} : \tilde{c} \in \mathrm{Crit}(T)\}} \leq |T'(x)| \leq e^{\theta}\ e^{-\alpha k_c(x)} \tag{2.1}$$

for every $x \in \overline{\mathbb{C}}$ and every critical point $c \in \overline{\mathbb{C}}$. Moreover, if $x \in J$ then (2.1) holds where the maximum is taken only over $\tilde{c} \in \mathrm{Crit}(T) \cap J(T)$.

This lemma will be needed in Section 3; here we need only the upper estimate.

Roughly speaking, the aim of this section is to estimate the sum of the distances of $T^j(x), j = 0, 1, ..., n,$ from a critical point $c \in J(T)$ in the logarithmic scale.

In [9, Section 2] the following simple fact was proved:

Lemma 2.2. (Rule I) There exists a constant $Q > 0$ such that for every $x \in \overline{\mathbb{C}}, c \in \mathrm{Crit}(T) \cap J(T), n \geq 1$

$$\min(k_c(x), k_c(T^n(x))) < Qn$$

In particular $\mathrm{dist}(T^n(c), c) > e^{-Qn}$.

Here we shall prove a stronger version (using this lemma one give a simplified proof for the positivity of Ljapunov exponents in [9]):
Lemma 2.3. (Rule II) There exists a constant $Q > 0$ such that if $c \in J(T)$ is a critical point of T, $n \geq 1$ is an integer, and if $x \in J$ satisfies

$$k_c(T^j(x)) \leq k_c(T^n(x)) \quad \text{for every} \quad j = 1, 2, ..., n-1, \tag{2.2}$$

then

$$\min\{k_c(x), k_c(T^n(x))\} + \sum_{j=1}^{n-1} k_c(T^j(x)) \leq Qn. \tag{2.3.}$$

Proof. The proof is by induction over n. For $n = 1$ the statement is Lemma 2.2. The procedure for the induction step will be as follows: Given $x, T(x), ..., T^n(x)$ satisfying (2.2) we shall decompose this string into two blocks: (a) $x, T(x), ..., T^m(x), m \leq n$ for which we shall prove (2.3); (b) $T^m(x), ..., T^n(x)$ for which we can apply the induction hypothesis. Summing these two estimates we prove (2.3) for $x, T(x), ..., T^n(x)$. It will be seen that it sufficies to take $Q = \alpha^{-1}(\log 2 + \theta + 1)$.

Let $k' = \min(k_c(x), k_c(T^n(x)))$ and $B = B(c, ae^{-(k'-1)})$.
If $k' \geq 1$, let $1 \leq m \leq n$ be the first positive integer such that

$$k_c(T^m(x)) - \inf\{k_c(T^m(z)) : z \in B\} > 1 \tag{i}$$

or

$$k_c(T^m(x)) \geq k'. \tag{ii}$$

If $k' = 0$ set

$$m = 1. \tag{iii}$$

In all these cases the sequence $y = T^m(x), T(y), .., T^{n-m}(y)$ satisfies the assumption (2.2) automatically and, moreover, $k_c(y) = \min(k_c(y), k_c(T^{n-m}(y)))$.
Hence by the induction hypothesis

$$\sum_{j=m}^{n-1} k_c(T^j(x)) \leq Q(n-m). \tag{2.4}$$

Assume $k' \geq 1$ (case (i) or (ii)). By definition of m we have for every $0 < j < m$, and for every $z \in B$, $k_c(T^m(x)) \leq k_c(T^m(z)) + 1$. Hence by Lemma 2.1 it follows that

$$|(T^{m-1})'(T(z))| \leq e^{(m-1)\theta} e^{-\alpha \sum_{j=1}^{m-1} (k_c(T^j(x))-1)}.$$

Using also $|T'(z)| \leq e^{\theta} e^{-\alpha(k'-1)}$ we obtain

$$\frac{\mathrm{diam} T^m(B)}{\mathrm{diam} B} \leq e^{m\theta + m - \alpha(k' + \sum_{j=1}^{m-1} k_c(T^j(x)))}. \tag{2.5}$$

In case (i) but not (ii) it follows that

$$\mathrm{diam} T^m(B) \geq a(e^{-(k_c(T^m(x))-1)} - e^{-k_c(T^m(x))}) \geq a(e^{-(k'-1)} - e^{-k'}) = ae^{-k'}(e-1).$$

This together with (2.5) gives

$$\frac{e-1}{2e} \leq e^{m(\theta+1) - \alpha(k' + \sum_{j=1}^{m-1} k_c(T^j(x)))}$$

hence

$$k' + \sum_{j=1}^{m-1} k_c(T^j(x)) \leq \alpha^{-1}(m(\theta+1) + \log 2). \tag{2.6}$$

In the case (ii) we also obtain

$$\mathrm{diam} T^m(B(c, ae^{k'-1})) \geq \frac{a}{2}(e^{-(k'-1)} - e^{-k'}).$$

Otherwise $T^m(B) \subset B$ and for $a < \mathrm{diam}\, \overline{\mathbb{C}}$ the set $\overline{\mathbb{C}} \setminus B$ contains at least 3 points. So the family of functions $(T^{tm}|_B)_{t=1,2,\ldots}$ is normal, which contradicts $c \in J(T)$. Consequently (2.6) holds in this case also.

Finally in the case (iii), we have (2.6) trivially (for $m = 1$).

Defining $Q = \alpha^{-1}(\log 2 + \theta + 1)$, (2.6) and (2.4) imply

$$k' + \sum_{j=1}^{n-1} k_c(T^j(x)) \leq Qn.$$

This finishes the proof. ♣

§3. Hölder continuity

In this section we prove that for every $0 < \alpha \leq 1$ there exists $\tau > 0$ such that

$$\sup_{n\geq 0}\{||\mathcal{L}_\phi^n(\psi)||_\tau\} < \infty \qquad \text{for all } \ \psi \in \mathcal{H}_\alpha, \tag{3.1}$$

Remark 3.1. More precisely, one may choose $\tau = \min(\tau', \frac{1}{\nu}\alpha)$, where τ' depends only on T and ϕ, and ν is the maximal multiplicity of iterates of T at critical points in J. Observe that the estimate $\tau \leq \frac{1}{\nu}\alpha$ is sharp because for $\psi(z) = \text{dist}(z, c)^\alpha$ and z close to $c \in \text{Crit}(T) \cap J(T)$ we have $|\mathcal{L}(\psi)(z) - \mathcal{L}(\psi)(c)| \geq \text{Const. } \text{dist}(z, c)^{\frac{\alpha}{\nu(c)}}$ where $\nu(c)$ denotes the multiplicity of T at c.

Property (3.1) will be checked using the following sufficient condition obtained in [8, Remark (3) after Proposition 3]:

Lemma 3.2. Let $L \geq 1$ and $\rho > 0$ be such that for every $x \in J(T)$, $\varepsilon > 0$, $n \geq 0$, and every connected component V of $T^{-n}(B(x, \varepsilon))$ one has $\text{diam}(V) \leq L^n\varepsilon^\rho$. Then for every α there exists τ such that (3.1) holds.

Formally it is proved only in [8] that the density function $h = d\mu/dm = \lim \mathcal{L}_\phi^n(1\!\!1)$ is Hölder-continuous, but the proof given there remains unchanged for Lemma 3.2. Observe that $\lim_{n\to\infty} \mathcal{L}_\phi^n(\psi) = \lim_{n\to\infty} \mathcal{L}_\phi^n(1\!\!1)$ does not depend on the exponent of Hölder-continuity of ψ, although this exponent for $\mathcal{L}_\phi^n(\psi)$ depends on that of ψ.

We first note the following elementary fact; its proof is left to the reader:

Lemma 3.3. There exists $M > 0$ such that for every connected subset $F \subset \overline{\mathbb{C}}$

$$\text{diam}(T(F)) \geq M \sup\{|T'(x)| : x \in F\}\text{diam}(F)$$

and

$$\text{diam}(T(F)) \geq M \text{diam}(F)^r$$

where r is the maximal multiplicity of T at critical points in $J(T)$.

So, our aim is to prove the following.

Lemma 3.4. There exist $L \geq 1$ and $\rho, \varepsilon_0 > 0$ such that for every $x \in J(T)$, $0 < \varepsilon \leq \varepsilon_0$, $n \geq 0$, and every connected component V of $T^{-n}(B(x,\varepsilon))$ one has $\mathrm{diam}(V) \leq L^n \varepsilon^\rho$.

Proof. Fix $x \in J(T)$ and fix $c \in \mathrm{Crit}(T) \cap J(T)$ for the moment.

Let $q(c) = t_1$ denote that index in $\{0, ..., n\}$ for which $k_c(T^t(x))$ attains its maximum (recall that $k_c(T^t(x)) = \infty$ is even possible, if $c = T^t(x)$, but there exists at most one such t). Recursively, define t_l to be that index in $\{t_{l-1}+1, ..., n\}$ where $k_c(T^t(x))$ attains its maximum. This procedure terminates after finitely many steps, say with t_u.

We decompose the trajectory $x, T(x), ..., T^n(x)$ into pieces (with overlapping ends)

$$\big(x, \dots, T^{t_1}(x)\big), \big(T^{t_1}(x), \dots, T^{t_2}(x)\big), \ \dots, \ \big(T^{t_{u-1}}(x), \dots, T^{t_u}(x)\big)$$

Observe that these pieces satisfy the assumptions of Lemma 2.3 and

$$k_c(T^{t_1}(x)) \geq k_c(T^{t_2}(x)) \geq \dots \geq k_c(T^{t_{u-1}}(x)) \geq k_c(T^{t_u}(x)).$$

Applying Lemma 2.3 we obtain

$$\sum_{j=0}^{t_1-1} k_c(T^j(x)) + \sum_{j=t_1+1}^{n} k_c(T^j(y)) \leq Qn. \tag{3.2}$$

Set $N = \#\mathrm{Crit}(T) \cap J$ and order the critical points in J by $c_1, c_2, \dots, c_N$ so that $0 \leq q_1 := q(c_1) \leq q_2 := q(c_2) \leq \dots \leq q_N := q(c_N)$. Setting

$$k(x) = \max_{c \in \mathrm{Crit}(T) \cap J(T)} k_c(x)$$

we get by (3.2)

$$\sum k(T^j(y)) \leq NQn, \tag{3.3}$$

where the sum is taken over all integers $j \in \{0, 1, 2, \ldots, n\} \backslash \{q_l : 1 \le l \le N\}$. Setting also $q_{N+1} = n$, for any $2 \le j \le N+1$ such that $q_j - q_{j-1} \ge 1$ let

$$\Pi_j = M^{q_j - q_{j-1}} \ e^{-\theta(q_j - q_{j-1} - 1)} \exp\left(-\alpha \sum_{i=q_{j-1}}^{q_j - 1} k(T^j(y))\right)$$

and

$$\Pi_1 = \mathrm{diam}(V) M^{q_1} \mathrm{e}^{-\theta q_1} \exp\bigl(-\alpha \sum_{i=0}^{q_1 - 1} k(T^j(y))\bigr).$$

Then it follows from Lemma 2.1 and Lemma 3.3 that

$$2\varepsilon = \mathrm{diam}(T^n(V)) \ge \bigl(\ldots\bigl(\bigl(\Pi_1^r \Pi_2\bigr)^r \Pi_3\bigr)^r \ldots \Pi_N\bigr)^r \Pi_{N+1}.$$

Hence applying (3.3) we can continue

$$\begin{aligned}
2\varepsilon &\ge \Pi_1^{r^N} \Pi_2^{r^{N-1}} \ldots \Pi_N^r \Pi_{N+1} \ge \bigl(\Pi_1 \Pi_2 \ldots \Pi_N \Pi_{N+1}\bigr)^{r^N} \\
&\ge \mathrm{diam}(V)^{r^N} (M \ e^{-\theta})^{r^N n} \exp\left(-\alpha r^N \sum_{j=0}^{N+1} \sum_{i=q_{j-1}}^{q_j - 1} k(T^i(y))\right) \\
&\ge \mathrm{diam}(V)^{r^N} \bigl(M \exp(-(\theta + \alpha N Q))\bigr)^{n r^N}
\end{aligned}$$

Therefore, setting $L = 2^{r^{-N}} M^{-1} \exp(\theta + \alpha N Q)$, we get $\mathrm{diam}(V) \le L^n \varepsilon^{r^{-N}}$. The proof is finished. ♣

As an immediate consequence of Lemma 3.2 (as mentioned above) we get the following:

Corollary 3.5. The Radon-Nikodym derivative of the equilibrium measure μ of the potential ϕ with respect to the $\exp[P(T, \phi) - \phi]$-conformal measure m, is a Hölder-continuous function.

§4. The rate of convergence of the Perron-Frobenius-Ruelle operator.

As in the introduction let μ denote the equilibrium state of a Hölder-continuous function $\phi : J(T) \to I\!R$ with $P(T, \phi) > \sup_{z \in J(T)} \phi(z)$. The associated conformal measure is denoted by m and the associated equivalent invariant measure by μ.

It has been proved in [3] that there exist a measurable Markov partition α of $J(T)$ and numbers $0 < \lambda < 1$ and $C > 0$ such that for $A \in \alpha$, $T(A) = J(T)$ μ a.e. and for all $n \geq 1$

(4.1)

$$\mu\Big(\bigcup\{A \in \bigvee_{j=0}^{n-1} T^{-j}(\alpha) : \mathrm{diam}(T^k(A)) > C\lambda^{n-k} \text{ for some } k = 0, 1, 2, \ldots, n\}\Big) < 1/10$$

Let

$$\mathcal{L}_\phi(\psi)(x) = \sum_{T(y)=x} \psi(y) \exp(\phi(y) - P(T, \phi))$$

denote the Perron-Frobenius-Ruelle operator (transfer operator) acting on $C(J(T))$. Note that $\mathcal{L}_\phi(h) = h$. Our aim in this section is to prove the following.

Theorem 4.1. Let $\psi : J(T) \to I\!R$ is a Hölder-continuous function. Then there are constants $L, \theta > 0$ such that

$$||\mathcal{L}_\phi^n(\psi) - \ h \int \psi \, dm||_\infty \leq L \exp[-\theta\sqrt{n}]$$

for all $n \geq 0$.

Let us begin the proof of this theorem defining a new type of Perron-Frobenius-Ruelle operator $\mathcal{L}_0$ by the formula $\mathcal{L}_0(\psi) = \mathcal{L}_\phi(\frac{h}{h \circ T}\psi)$. Note that $\mathcal{L}_0(1) = 1$ and that $\mathcal{L}_0$ preserves the space of continuous functions, since h is a continuous, nowhere vanishing function. Moreover, in view of Corollary 3.5 and Lemma 3.2

$$C_1 = \sup_{n \geq 0}\{||\mathcal{L}_0^n(\psi)||_\tau\} < \infty \tag{4.2}$$

where $\tau > 0$ depends only on the map T and the functions ϕ and ψ. Since for every $x \in J$ and every integer $n \geq 1$

$$\mathcal{L}_0^n(\frac{\psi}{h})(x) = \frac{1}{h(x)}\mathcal{L}_\phi^n(\psi)(x),$$

it follows that

$$\begin{aligned}(\mathcal{L}_\phi^n(\psi) - h\int \psi\, dm)(x) &= h(x)\mathcal{L}_0^n\left(\frac{\psi}{h} - \int \psi\, dm\right)(x) \\ &= h(x)\mathcal{L}_0^n\left(\frac{\psi}{h} - \int \frac{\psi}{h}\, d\mu\right)(x) \\ &= h(x)\left(\mathcal{L}_0^n\left(\frac{\psi}{h}\right) - \int \frac{\psi}{h}\, d\mu\right)(x),\end{aligned}$$

we get $||\mathcal{L}_\phi^n(\psi) - h\int \psi\, dm||_\infty \leq ||h||_\infty \left|\left|\mathcal{L}_0^n\left(\frac{\psi}{h}\right) - \int \frac{\psi}{h}\, d\mu\right|\right|_\infty$ and therefore the proof of Theorem 4.1 reduces to the proof of its version for the operator $\mathcal{L}_0$ which is formulated below:

Lemma 4.2. Let $\psi : J(T) \to I\!R$ is a Hölder-continuous function with exponent τ. Then there are constants $C_2, \theta > 0$ such that

$$||\mathcal{L}_0^n(\psi) - \int \psi\, d\mu||_\infty \leq C_2 \exp[-\theta\sqrt{n}]$$

for all $n \geq 0$.

Let us begin the proof of this statement with the following lemma.

Lemma 4.3. If $\Delta \geq ||\psi - \int \psi\, d\mu||_\infty$, then $\mu(\{x : \psi(x) - \int \psi\, d\mu \leq \Delta/4\}) \geq 1/5$.

Proof. Since there is nothing to prove for $\Delta = 0$, suppose that $\Delta > 0$. Let $A = \{x : \psi(x) - \int \psi\, d\mu \leq \Delta/4\}$ and let A^c be the complement of A. We have $0 = \int(\psi - \int \psi\, d\mu)\, d\mu = \int_A(\psi - \int \psi\, d\mu)\, d\mu + \int_{A^c}(\psi - \int \psi\, d\mu)\, d\mu \geq -\Delta\mu(A) + \mu(A^c)\Delta/4 = \Delta(-\mu(A) + (1 - \mu(A)/4) = (1 - 5\mu(A))\Delta/4$. Hence $1 - 5\mu(A) \leq 0$ which finishes the proof. ♣

For each $n \geq 0$ let α_b^n be the collection of all elements of the partition $\alpha^n = \bigvee_{j=0}^{n-1} T^{-j}(\alpha)$ defined in (4.1) and let $\alpha_g^n = \alpha^n \setminus \alpha_b^n$. Recall that $\mathcal{L}_0^n(\psi)(x) = \sum_{y \in T^{-n}(x)} g_n(y)\psi(y)$, where

$$g_n(y) = \frac{h(y)}{h(x)} \exp\bigl(-\mathrm{P}(T,\phi)n + \phi(y) + \phi(T(y)) + ... + \phi(T^{n-1}(y))\bigr).$$

A straightforward computation shows (see for example [3]) that there exists $C_2 > 0$ such that

$$\frac{g_n(x)}{g_n(y)} \leq C_2 \tag{4.3}$$

for all $n \geq 1$, all $A \in \alpha_g^n$, and all $x, y \in A$. Let $\gamma = 1 - 1/(20C_2)$. Given $\Delta > 0$ let $n = n(\Delta) \geq 0$ be the smallest positive integer $\geq (\log \Delta - \log(4C_1C^\tau))/\tau \log \lambda$. We shall prove the following.

Lemma 4.4. If $||\psi - \int \psi \, d\mu||_\infty \leq \Delta$, then $||\mathcal{L}_0^{n(\Delta)}(\psi) - \int \psi \, d\mu||_\infty \leq \gamma\Delta$.

Proof. Let $G = \{x : \psi(x) - \int \psi \, d\mu \leq \Delta/4\}$. For each $n \geq 1$ define $\alpha_G^n = \{A \in \alpha_g^n : A \cap G \neq \emptyset\}$. First we shall show that if $x \in \bigcup \alpha_G^n$, then

$$\psi(x) - \int \psi \, d\mu \leq \Delta/2. \tag{4.4}$$

Indeed, by definition of $n(\Delta)$ we have $C_1 C^\tau \lambda^{\tau n(\Delta)} \leq \Delta/4$. Therefore if $y \in A \cap G$ and $x \in A$, then we get $\psi(x) - \int \psi \, d\mu = \psi(x) - \psi(y) + \psi(y) - \int \psi \, d\mu \leq C_1|x - y|^\tau + \Delta/4 \leq C_1(C\lambda^{n(\Delta)})^\tau + \Delta/4 \leq \Delta/2$. The proof of (4.4) is finished.

Set now $n = n(\Delta)$ and for every $x \in J(T)$ define $G_n(x) = T^{-n}(x) \cap \bigcup \alpha_G^n$

and $B_n(x) = T^{-n}(x) \setminus G_n(x)$. Using (4.4) we can write

$$\begin{aligned}
\mathcal{L}_0^n(\psi)(x) - \int \psi\, d\mu &= \sum_{y \in T^{-n}(x)} g_n(y)\big(\psi(y) - \int \psi\, d\mu\big) \\
&= \sum_{y \in G_n(x)} g_n(y)\big(\psi(y) - \int \psi\, d\mu\big) + \sum_{y \in B_n(x)} g_n(y)\big(\psi(y) - \int \psi\, d\mu\big) \\
&\le \frac{\Delta}{2} \sum_{y \in G_n(x)} g_n(y) + \Delta \sum_{y \in B_n(x)} g_n(y) \\
&= \Delta \left(\frac{1}{2} \sum_{y \in G_n(x)} g_n(y) + 1 - \sum_{y \in G_n(x)} g_n(y) \right) \\
&= \Delta \left(1 - \frac{1}{2} \sum_{y \in G_n(x)} g_n(y) \right) \qquad (4.5)
\end{aligned}$$

It follows from (4.1) and Lemma 4.3 that $\mu\big(\bigcup \alpha_G^n\big) \ge 1/5 - 1/10 = 1/10$. For every $z \in J(T)$ and every $A \in \alpha^n$, denote by $\phi_A^{(n)}(z)$ the uniquely defined element of the set $A \cap T^{-n}(z)$. Using (4.3) we can write $\mu\big(\bigcup \alpha_G^n\big) = \sum_{A \in \alpha_G^n} \int g_n(\phi_A^{(n)}(z))\, d\mu(z) \le \sum_{A \in \alpha_G^n} C_2 g_n(\phi_A^{(n)}(x)) = C_2 \sum_{y \in G_n(x)} g_n(y)$ and thus $\sum_{y \in G_n(x)} g_n(y) \ge 1/(10C_2)$. Combining this and (4.5) we get $\mathcal{L}_0^n(\psi)(x) - \int \psi\, d\mu \le \Delta(1 - 1/(20C_2))$. Replacing ψ by $-\psi$ yields $\mathcal{L}_0^n(\psi)(x) - \int \psi\, d\mu \ge -\Delta(1 - 1/(20C_2))$ which finishes the proof of Lemma 4.4. ♣

Proof of Lemma 4.2. For every integer $j \ge 0$ define $\Delta_j = \gamma^j ||\psi - \int \psi\, d\mu||_\infty$ and then inductively the sequence $\{n_j : j \ge 0\}$ setting $n_0 = 0$ and $n_{j+1} = n_j + n(\Delta_j)$. It follows from Lemma 4.4 that

$$||\mathcal{L}_0^{n_j}(\psi) - \int \psi\, d\mu||_\infty = ||\mathcal{L}_0^{n(\Delta_{j-1})}(\mathcal{L}_0^{n_{j-1}}(\psi)) - \int \mathcal{L}_0^{n_{j-1}}(\psi)\, d\mu||_\infty \le \gamma \Delta_{j-1} = \Delta_j \qquad (4.6)$$

By the definition of the integers n_j we have

$$\frac{\log \Delta_j - \log(4C_1 C^\tau)}{\tau \log \lambda} \le n_{j+1} - n_j \le \frac{\log \Delta_j - \log(4C_1 C^\tau)}{\tau \log \lambda} + 1$$

Setting $C_3 = \log\gamma/\log\lambda > 0$ and $C_4 = (\log(||\psi - \int \psi\, d\mu||_\infty - \log(4C_1 C^\tau))/\tau \log\lambda$, rewrite these inequalities in the form $C_3 j + C_4 \le n_{j+1} - n_j \le C_3 j + C_4 + 1$. Thus, summing up, we get for all $k \ge 1$

$$C_3 \sum_{j=0}^{k-1} j + C_4 k \le n_k \le C_3 \sum_{j=0}^{k-1} j + (C_4 + 1)k,$$

or equivalently

$$C_3 \frac{k(k-1)}{2} + C_4 k \le n_k \le C_3 \frac{k(k-1)}{2} + (C_4 + 1)k.$$

Hence there exists a constant $C_5 > 0$ such that

$$C_5^{-1} k^2 \le n_k \le C_5 k^2. \tag{4.7}$$

In view of this and (4.6), $\log\big(||\mathcal{L}_0^{n_k}(\psi) - \int \psi\, d\mu||_\infty\big) \le \log(\Delta_k) = k \log\gamma + \log(||\psi - \int \psi\, d\mu||_\infty) \le \sqrt{C_5} \log\gamma \sqrt{n_k} + \log(||\psi - \int \psi\, d\mu||_\infty)$. Thus

$$||\mathcal{L}_0^{n_k}(\psi) - \int \psi\, d\mu||_\infty \le ||\psi - \int \psi\, d\mu||_\infty) \exp[\sqrt{C_5} \log\gamma \sqrt{n_k}]. \tag{4.8}$$

Take now any $n \ge n_1$. Then there exists $k \ge 1$ such that $n_k \le n \le n_{k+1}$. Hence by (4.7), $\sqrt{n_k} \ge \sqrt{C_5^{-1}} k \ge \frac{1}{2}\sqrt{C_5^{-2}}\sqrt{C_5}(k+1) \ge \frac{1}{2}\sqrt{C_5^{-2}} n_{k+1} \ge \frac{1}{2}\sqrt{C_5^{-1}} n$. Therefore, it follows from (4.8) that $||\mathcal{L}_0^{n}(\psi) - \int \psi\, d\mu||_\infty \le ||\mathcal{L}_0^{n_k}(\psi) - \int \psi\, d\mu||_\infty \le \sqrt{C_5} \log\gamma \sqrt{n_k} + \log(||\psi - \int \psi\, d\mu||_\infty) \le \frac{1}{2} \log\gamma \sqrt{n} + \log(||\psi - \int \psi\, d\mu||_\infty)$. Therefore $||\mathcal{L}_0^{n}(\psi) - \int \psi\, d\mu||_\infty \le ||\psi - \int \psi\, d\mu||_\infty \exp[-\tilde\theta \sqrt{n}]$ for every $n \ge n_1$, where $\tilde\theta = -\frac{1}{2}\log\gamma$. Hence, taking $\theta > 0$ large enough the proof of Lemma 4.2 is completed. ♣

§5. The central limit theorem

Let $\phi : J \to I\!R$ be a Hölder-continuous potential with

$$P(T, \phi) > \sup_{z \in J} \phi(z).$$

Let m denote the unique $\exp[P(f, \phi) - \phi]$-conformal measure and $\mu \sim m$ the unique invariant measure equivalent to m.

In this section we proof the central limit theorem for Hölder-continuous functions with respect to the equilibrium measure μ. Since we want to use Gordin's theorem (see [5]) we need to consider the natural extension of T:

$$T^* : J^* := \{(x_k)_{k \in -I\!N} : T(x_k) = x_{k+1} \quad (k \leq -1)\} \to J^*,$$

where

$$T^*((x_k)_{k \in -I\!N}) = (T(x_k))_{k \in -I\!N}.$$

Then the σ-field $\mathcal{B}$ of Borel sets in J defines a σ-field $\mathcal{M}_0$ in J^* by

$$\mathcal{M}_0 = \pi^{-1}\mathcal{B},$$

where π denotes the projection of J^* onto the first coordinate. It is clear that $(T^*)^{-1}\mathcal{M}_0 \subset \mathcal{M}_0$. Denote μ^* the natural extension of μ to J^* and U the unitary operator induced by T^* on the space $L_2(\mu^*)$. Let $L_2(\mathcal{M}_0)$ denote the space of square integrable functions which are measurable with respect to $\mathcal{M}_0$ and set

$$H_k := U^k\big(L_2(\mathcal{M}_0)\big) \qquad (k \in \boldsymbol{Z}).$$

Then $H_{k+1} \subset H_k$ for every $k \in \boldsymbol{Z}$. Also, H_0 is isomorphic to $L_2(\mu)$.

Lemma 5.1. The dual operator T^* of the restriction of U to H_0 is given by

$$T^*(g) = \frac{\mathcal{L}_\phi(gh)}{h} \qquad m-\text{a.e.}$$

for $g \in H_0$, where h denotes the density $\frac{d\mu}{dm}$ as before.
Proof. If g is $\mathcal{M}_0$-measurable, then

$$< T^*(g), \psi > = < g, U(\psi) > = \int g(\psi \circ T)\, d\mu$$
$$= \int gh(\psi \circ T)\, dm = \int \mathcal{L}_\phi(gh)\ \psi\, dm$$
$$= \int \frac{\mathcal{L}_\phi(gh)}{h}\ \psi\, d\mu = < \frac{\mathcal{L}_\phi(gh)}{h}, \psi >,$$

since

$$\mathcal{L}_\phi(gh(\psi\circ T))(z) = \sum_{T(y)=z} g(y)h(y)\psi(T(y))\exp[\phi(y)-P(T,\phi)]$$
$$= \psi(z)\sum_{T(y)=z} g(y)h(y)\exp[\phi(y)].$$

♣

Lemma 5.2. The operator $U^k\circ(T^*)^k$ is the orthogonal projection of H_0 onto H_k for any $k\geq 0$.
Proof. Let $g\in H_0$. We only need to show that for

$$\tilde{\psi} = \psi\circ T^k$$

we have

$$< g - U^k((T^*)^k(g)), \tilde{\psi} >= 0.$$

But this follows immediately from

$$\int (U^k((T^*)^k(g)))\cdot(U^k(\psi))\,d\mu^* = \int ((T^*)^k(g))\cdot\psi\,d\mu$$
$$= \int \mathcal{L}_\phi\big[((T^*)^{k-1}(g))\cdot h\big]\cdot\frac{1}{h}\cdot\psi\,h\,dm$$
$$= \int \mathcal{L}_\phi\big[((T^*)^{k-1}(g))\cdot h\cdot U(\psi)\big]\,dm$$
$$= \int ((T^*)^{k-1}(g))\cdot U(\psi)\,d\mu$$
$$= \ldots = \int g\cdot U^k(\psi)\,d\mu.$$

♣

Recall Gordin's theorem in the present set up: Denote P_k ($k\in\boldsymbol{Z}$) the orthogonal projection of $L_2(\mu^*)$ onto H_k. If

$$\sum_{k\geq 0}\|P_k(g)\|_2 + \|g-P_{-k}(g)\|_2 < \infty,$$

then there exists $\sigma \geq 0$ such that

$$\lim_{n\to\infty} \frac{1}{n} \int \left(\sum_{k=0}^{n-1} U^k(g) \right)^2 d\mu^* = \sigma^2,$$

and if $\sigma^2 > 0$, then for any $t \in I\!R$

$$\mu^* \left(\{ z \in J^* : \frac{1}{\sqrt{n\sigma^2}} \sum_{k=0}^{n-1} U^k(g)(z) \leq t \} \right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp[-u^2/2]\, du.$$

In this case we say that g satisfies the central limit theorem.

We now can prove our result. For an integrable function g denote $\mu(g) = \int g\, d\mu$.

Theorem 5.3 Every Hölder continuous function $g : J \to I\!R$ satisfies the central limit theorem:

There exists $\sigma \geq 0$ such that

$$\lim_{n\to\infty} \frac{1}{n} \int \left(\sum_{k=0}^{n-1} (g \circ T^k - \mu(g)) \right)^2 d\mu = \sigma^2,$$

and if $\sigma^2 > 0$, then for any $t \in I\!R$

$$\mu \left(\{ z \in J : \frac{1}{\sqrt{n\sigma^2}} \sum_{k=0}^{n-1} \big(g(T^k(z)) - \mu(g) \big) \leq t \} \right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp[-u^2/2]\, du.$$

Proof: First note that g is a bounded, $\mathcal{M}_0$- measurable function, hence it may be considered to belong to H_0. We also may assume that $\mu(g) = 0$. Then $P_{-k}(g) = g$, and it is left to show that

$$\sum_{k\geq 0} \|P_k(g)\|_2 < \infty.$$

Since by Lemma 5.2 $P_k = U^k(T^*)^k$, it suffices to show that

$$\sum_{k\geq 0} \|U^k((T^*)^k(g))\|_2 < \infty.$$

We compute

$$\|U^k((T^*)^k(g))\|_2 = \int (U^k((T^*)^k(g)))^2\, d\mu = \int ((T^*)^k(g)) \cdot ((T^*)^k(g))\, d\mu$$
$$= \int (U^k((T^*)^k(g))) \cdot g\, d\mu \le \|g\|_\infty \int |(T^*)^k g|\, d\mu = \|g\|_\infty \int \left| \frac{\mathcal{L}_\phi((T^*)^{k-1}(g) \cdot h)}{h} \right| d\mu$$
$$\le \|g\|_\infty \int \left|\mathcal{L}_\phi((T^*)^{k-1} g \cdot h)\right| dm = \|g\|_\infty \int \left|\mathcal{L}_\phi[\mathcal{L}_\phi((T^*)^{k-2}(g) \cdot h)]\right| dm = ...$$
$$= \|g\|_\infty \int |\mathcal{L}_\phi^k(g \cdot h)|\, dm.$$

Now we apply Theorem 4.1 to the function $\psi = g \cdot h$. We have

$$\int \psi dm = \int g\, d\mu = 0$$

and hence

$$\|\mathcal{L}_\phi^k(g \cdot h)\|_\infty \le C_2 \exp[-\theta\sqrt{n}].$$

It follows that

$$\|U^k((T^*)^k(g))\|_2 \le C_2 \|g\|_\infty \exp[-\theta\sqrt{n}].$$

This proves the theorem. ♣

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