THERMODYNAMIC FORMALISM FOR COARSE EXPANDING DYNAMICAL SYSTEMS

TUSHAR DAS, FELIKS PRZYTYCKI, GIULIO TIOZZO, MARIUSZ URBAŃSKI, AND ANNA ZDUNIK

ABSTRACT. We consider a class of dynamical systems, which we call *weakly coarse expanding*, which is a generalization to the postcritically infinite case of *expanding Thurston* maps as discussed by Bonk–Meyer and is closely related to *coarse expanding conformal* systems as defined by Haïssinsky–Pilgrim. We prove existence and uniqueness of equilibrium states for a wide class of potentials, as well as statistical laws such as a central limit theorem, law of iterated logarithm, exponential decay of correlations and a large deviation principle. Further, if the system is defined on the 2-sphere, we prove all such results even in presence of periodic (repelling) branch points.

1. INTRODUCTION

The study of dynamical systems through thermodynamic formalism goes back to Sinai [28], Bowen [5], and Ruelle [26]. The paradigmatic example in this theory is the one-sided shift map on the space of sequences on a finite alphabet. This space is endowed with a natural metric under which the shift is uniformly expanding. This feature is essential in most applications: indeed, various systems which possess a certain degree of expansion can be encoded by a shift of finite type (see [24]), and this technique has allowed to establish statistical laws for a large class of dynamical systems.

For instance, several authors (see [23] and references therein) have addressed the case of rational maps, i.e. holomorphic endomorphisms of the 2-sphere, and established statistical laws for them.

A related class of examples of continuous self-maps of the 2-sphere is represented by *expanding Thurston maps*. Such maps are postcritically finite, and can be described by a finite set of "cut and fold" rules [6]; note that they need not be Thurston equivalent to holomorphic maps. These maps are discussed by Bonk–Meyer [4], who among other things construct the measure of maximal entropy. In a series of papers ([18], [19], [20], [21]), Li works out the thermodynamic formalism for expanding Thurston maps, with respect to the visual metric.

In a similar vein, Haïssinsky–Pilgrim [15] define more generally the concept of *coarse expanding conformal* (cxc) system, developing in an axiomatic way the notion of expansion for maps of general metric spaces. In particular, they prove for cxc systems existence and uniqueness of the measure of maximal entropy.

The goal of this paper is to develop the thermodynamic formalism (in particular, prove existence and uniqueness of equilibrium states and statistical laws) for a general class of dynamical systems, which we call *weakly coarse expanding*. These systems are continuous finite branched coverings of locally connected topological spaces which are expanding in a weak metric sense, and generalize most cases discussed by [4] and [15]. In particular, they need not be postcritically finite, and the metric we consider need not be the visual metric, but it could be more generally an exponentially contracting metric (see Section 2 for the definitions). Let us stress that neither conformality, nor holomorphy or smoothness is assumed in this paper. In particular, periodic branch points may be repelling despite the local degree at them being bigger than 1.

Our main results are contained in the following two theorems.

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Theorem 1.1. *Let $f : W_1 \to W_0$ be a weakly coarse expanding dynamical system without periodic critical points, let X be its repellor, ρ an exponentially contracting metric on X compatible with the topology, and let $\varphi : (X, \rho) \to \mathbb{R}$ be a Hölder continuous function. Then:*

(1) there exists a unique equilibrium state μ_φ for φ on X.
Let $\psi : (X, \rho) \to \mathbb{R}$ be a Hölder continuous observable, and denote

$$S_n\psi(x) := \sum_{k=0}^{n-1} \psi(f^k(x)).$$

Then there exists the finite limit

$$\sigma^2 := \lim_{n\to\infty} \frac{1}{n} \int_X \left(S_n\psi(x) - n \int \psi \, d\mu_\varphi \right)^2 d\mu_\varphi \geq 0$$

such that the following statistical laws hold:

(2) (Central Limit Theorem, CLT) If $\sigma > 0$, we have for any $a < b$

$$\mu_\varphi \left(\left\{ x \in X \ : \ \frac{S_n\psi(x) - n\int_X \psi \, d\mu_\varphi}{\sqrt{n}} \in [a, b] \right\} \right) \longrightarrow \frac{1}{\sqrt{2\pi\sigma^2}} \int_a^b e^{-t^2/2\sigma^2} \, dt$$

as $n \to \infty$. If $\sigma = 0$, one has convergence in probability to the Dirac δ-mass at 0.

(3) (Law of Iterated Logarithm, LIL) For μ_φ-a.e. $x \in X$,

$$\limsup_{n\to\infty} \frac{S_n\psi(x) - n\int_X \psi \, d\mu_\varphi}{\sqrt{n \log\log n}} = \sqrt{2\sigma^2}.$$

(4) (Exponential Decay of Correlations, EDC) There exist constants $\alpha > 0$ and $C \geq 0$ such that for any μ_φ–integrable function $\chi : X \to \mathbb{R}$, for any β-Hölder function $\psi : X \to \mathbb{R}$, and for any $n \geq 0$,

$$\left| \int_X \psi \cdot (\chi \circ f^n) \, d\mu_\varphi - \int_X \psi \, d\mu_\varphi \cdot \int_X \chi \, d\mu_\varphi \right| \leq Ce^{-n\alpha} \|\underline{\chi}\|_1 \cdot \|\underline{\psi}\|_\beta,$$

where $\underline{\chi} := \chi - \int_X \chi \, d\mu_\varphi$, and $\| \cdot \|_\beta$ is the β-Hölder norm.

(5) (Large Deviations, LD) For every $t \in \mathbb{R}$, we have that

$$\lim_{n\to\infty} \frac{1}{n} \log \mu_\varphi \left(\left\{ x \in X : \operatorname{sgn}(t) S_n\psi(x) \geq \operatorname{sgn}(t) n \int_X \psi \, d\mu_{\varphi+t\psi} \right\} \right)$$
$$= -t \int_X \psi \, d\mu_{\varphi+t\psi} + P_{top}(\varphi + t\psi) - P_{top}(\varphi).$$

(6) Moreover, $\sigma = 0$ if and only if there exists a continuous $u : X \to \mathbb{R}$ such that

$$\psi - \int_X \psi \, d\mu_\varphi = u \circ f - u.$$

(7) Finally, $\mu_{\varphi_1} = \mu_{\varphi_2}$ if and only if there exist $K \in \mathbb{R}$ and a continuous $u : X \to \mathbb{R}$ such that

$$\varphi_1 - \varphi_2 = u \circ f - u + K.$$

In (6) *and* (7)*, the function u is Hölder continuous with respect to a visual metric.*

Just as in classical holomorphic dynamics, a special role is played by *critical points*, i.e. points where the map is not locally injective, also called *branch points*. A major source of difficulty in the study of weakly coarse expanding systems is the presence of periodic critical points in the repellor (indeed, those systems do not satisfy the [Degree] condition, hence they are not coarse expanding conformal in the sense of [15]).

Our second result addresses this issue in case the underlying space is an open subset of the 2-sphere (which is the case considered by [4] and [21]); there, our results also hold in the presence of periodic critical points as follows.

Theorem 1.2. *If $f : W_1 \to W_0$ is a weakly coarse expanding system and $W_0 \subseteq S^2$ is an open subset of the 2-sphere, with the Euclidean topology, then all claims (1)-(2)-(3)-(4)-(5)-(6)-(7) of Theorem 1.1 hold even if there are periodic critical points.*

Examples of weakly coarse expanding systems are given, other than in [4] and [15], in [16]; in particular, iterated function systems [12], skew products induced by some finitely generated semigroups of rational functions, in particular those given by Examples 17.1 and 17.2 in [1], maps on Sierpiński carpets and gaskets, as well as expanding *polymodials* ([2], [3]).

For holomorphic examples, note there is an abundance of non-hyperbolic, non-postcritically finite rational maps $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which have recurrent critical points in the Julia set $X = J(f)$ and satisfy backward exponential contraction (as in Theorem 2.12) for the Riemann metric ρ. This contraction condition is equivalent to the so-called *Topological Collet-Eckmann (TCE)* condition and is satisfied e.g. for a Lebesgue-positive measure set of non-hyperbolic recurrent real parameters c for the family $f_c(z) = z^2 + c$ (see [7], [30] and [23]). These maps are weakly coarse expanding but not topologically cxc, since the [Degree] condition fails because of recurrence of the critical point $z = 0$.

For rational maps $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ satisfying TCE, all the assertions (1)-(7) of Theorem 1.1 hold for all Hölder potentials $\varphi : J(f) \to \mathbb{R}$, see [17, Corollary 1.2] and [8]. In fact the claims (1)-(7) hold for all (not only TCE) rational maps f provided φ satisfies $P(f^n, S_n(\varphi)) > n \sup \varphi$ for some $n \in \mathbb{N}$. See [23, Section 3] for references.

Note that all results in Theorems 1.1 and 1.2 work for φ and ψ lying in the slightly more general class of *topologically Hölder* functions, as defined in Section 5.1. Let us also remark that our proofs follow a different, and in a way simpler, route than [21]: indeed, there the author directly applies the analytic methods of [24] on the metric space X. In the present paper, on the other hand, we first encode the dynamics of the system by a geometric coding tree, in the footsteps of [22], obtaining a semiconjugacy with the shift space; then, our statistical laws follow easily from the corresponding ones for the shift space. As for the second part (proof of Theorem 1.2), our technique consists in blowing up periodic critical points (and their grand orbit) to circles. The new space obtained is a Sierpiński carpet, and the dynamics becomes truly coarse expanding (without periodic critical points) on it, hence we can apply the same techniques of Theorem 1.1 to this new space, obtaining the desired results.

We have been informed by Peter Haïssinsky that he independently obtained a statement similar to Theorem 1.1.

The paper is organized as follows. The first part deals with the case without periodic critical points. In Section 2, we introduce the definition of weakly coarse expanding systems and construct the geometric coding trees. In Section 3, we prove that entropy does not drop under the projection from the symbolic space and in Section 4 we prove existence and uniqueness of equilibrium states as well as the statistical laws, proving Theorem 1.1 (1)-(5).

In the second part we deal with periodic critical points; in particular, in Section 5 we define a new space by blowing up the sphere along the grand orbit of every periodic critical (branch) point and produce, using Frink's lemma, an exponentially contracting metric, which we use to prove Theorem 1.2 (1)-(5). In Section 6 we deal with the cohomological equation, proving (6) and (7) of Theorem 1.1 and 1.2.

Finally, in Appendix A we provide an alternative proof by constructing explicitly an exponentially contracting metric on the blown up space, and in Appendix B we prove that the blown up space embeds into the sphere.

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2. Definitions

Let us start by introducing the fundamental definitions of the objects we are dealing with. We follow [15] in several places, though with some notable changes.

2.1. Finite branched coverings.

Let $f : Y \to Z$ be a continuous map between locally compact Hausdorff topological spaces. Note that such spaces satisfy the T_3 separation axiom (see e.g. [11], Chapter 3).

The *degree* of f is defined as

$$\deg(f) := \sup\{\#f^{-1}(z) \ : \ z \in Z\}.$$

Given a point $y \in Y$, the *local degree* of f at y is

$$\deg(f; y) := \inf_U \sup \#\{f^{-1}(z) \cap U \ : \ z \in f(U)\}$$

where U ranges over all open neighborhoods of y. A point y is *critical* if $\deg(f; y) > 1$.

Definition 2.1. *The map $f : Y \to Z$ is a* finite branched cover *of degree d if $\deg(f) = d < \infty$ and the two following conditions hold:*

(1) for any $z \in Z$,

$$\sum_{y \in f^{-1}(z)} \deg(f; y) = \deg(f)$$

(2) for any $y_0 \in Y$ there are compact neighborhoods U, V of y_0 and $f(y_0)$ such that

$$\sum_{y \in U, f(y) = z} \deg(f; y) = \deg(f; y_0)$$

for all $z \in V$.

We define the *branch set* as $B_f := \{y \in Y \ : \ \deg(f; y) > 1\}$, and the set of *branch values* as $V_f = f(B_f)$. Note that $Z \setminus V_f$ is the set of *principal values*, i.e. the values $z \in Z$ so that $\#f^{-1}(z) = d$.

Lemma 2.2 ([15], Lemma 2.1.2)**.** *A finite branched cover is open, closed, onto and proper. Furthermore, B_f and V_f are closed and nowhere dense.*

A simple corollary of the lemma is that f is a local homeomorphism away from critical points; i.e., for any $y \notin B_f$ there exists an open set U which contains y and such that $f : U \to f(U)$ is a homeomorphism.

Definition 2.3. *A topological space $\mathcal{X}$ is* strongly path connected *if for any countable subset $S \subset \mathcal{X}$, the space $\mathcal{X} \setminus S$ is path connected.*

Note that we need such an assumption: for instance, a tree is path connected but it becomes disconnected if you remove any point.

2.2. Coarse expanding dynamical systems.

Let W_1, W_0 be two locally compact, locally connected, strongly path connected, Hausdorff topological spaces, and suppose that W_1 is an open subset of W_0 and the closure of W_1 is compact.

We define a *system* as a triple (f, W_0, W_1), where W_0, W_1 satisfy the hypotheses above, and $f : W_1 \to W_0$ is a finite branched cover of degree $d \geq 2$.

Definition 2.4. *We define the* repellor X *of the system (f, W_0, W_1) as*

$$X := \bigcap_{n=0}^{\infty} f^{-n}(W_1).$$

By definition, $X = f^{-1}(X)$. Note that X is compact, by compactness of the closure of W_1. Moreover, we define the *post-branch set* as

$$P_f := X \cap \bigcup_{n>0} V_{f^n}.$$

Note that we do *not* take the closure of the post-branch set (differently from [15]).

Haïssinsky and Pilgrim [15, Section 2.2] give the following topological definition of expansion. Let $\mathcal{U}_0$ be a finite cover of X by connected, open subsets of W_1 whose intersection with X is not empty. For each n, we define $\mathcal{U}_n$ as the open cover whose elements are the connected components of $f^{-n}(U)$ where U belongs to $\mathcal{U}_0$. We shall call *pullbacks* of U the connected components of the preimages $f^{-n}(U)$.

Definition 2.5. *A system (f, W_0, W_1) satisfies the* [Expansion] *axiom if there exists a finite cover $\mathcal{U}_0$ of X such that the following holds: for any open cover $\mathcal{Y}$ of X by open subsets of W_0, there exists N such that for all $n \geq N$ each element of $\mathcal{U}_n$ is contained in some element of $\mathcal{Y}$. We say that $\mathcal{U}_n$ is* subordinated *to $\mathcal{Y}$.*

The following is useful. We denote as $\operatorname{Comp}_p U$ the connected component of U containing p.

Lemma 2.6. *Consider a system (f, W_0, W_1) satisfying the* [Expansion] *axiom with respect to a cover $\mathcal{U}$. Let $p \in X$ be a fixed point for f, where we allow p to be critical. Then the following holds:*

(1) There exist an open set $U \in \mathcal{U}$ containing p and $N \in \mathbb{N}$ such that $\operatorname{Comp}_p f^{-n}(U)$ lies in U for all $n \geq N$.

(2) Denoting $U_{n,p} := \operatorname{Comp}_p f^{-n}(U)$, we have

$$\bigcap_n U_{n,p} = \{p\}.$$

(3) (stronger than (1)) There exists N such that for every open neighborhood V of p there exists $n_V \in \mathbb{N}$ such that the open set $U = U_{n_V,p}$ satisfies $U \subseteq V$ and

$$\operatorname{Comp}_p f^{-n}(U) \subset U \quad \textit{for all } n \geq N.$$

(4) Moreover, for $U = U_{n_V+N,p}$ we obtain $\operatorname{Comp}_p f^{-n}(U) \subset U_{n_V,p} \subset V$ for all $n \geq 0$.

If p is periodic for f of period k, all claims still hold after replacing f by f^k.

In other words, (2) says that each periodic orbit in X is repelling (maybe not exponentially).

If W_0 is equipped with a metric, [Expansion] implies that the diameter of the cover $\mathcal{U}_n$ tends uniformly to zero as $n \to \infty$. The proof of Lemma 2.6 becomes immediate, since in Definition 2.5 of [Expansion] we can take covers by discs of arbitrarily small diameters. In a purely topological situation the proof is less obvious and illustrates the adequacy of the [Expansion] axiom.

First we provide the following.

Lemma 2.7. *Let (f, W_0, W_1) be a system satisfying the* [Expansion] *axiom, with a finite cover $\mathcal{Y}$ of X by open sets in W_0, with pullbacks $\mathcal{Y}_n, n = 0, 1, \ldots$. Then, given $p \in X$ and an element $Y(p)$ of $\mathcal{Y}_0$ containing p, we can modify $\mathcal{Y}$ to a new cover, still satisfying [Expansion], which contains $Y(p)$ and for which $Y(p)$ is the unique element containing p. As a consequence, in the modified cover for every n there is a unique element $Y_{n,p} \in \mathcal{Y}_n$ containing p.*

Proof. Choose an arbitrary $Y \in \mathcal{Y}_0$ containing p and subtract from all other $Y' \ni p$ a compact $K \subset Y$ containing p in its interior, which exists by the T_3 property of our topology on W_0. □

Proof of Lemma 2.6. Let us apply Lemma 2.7 by taking as $\mathcal{Y}$ the cover $\mathcal{U}$ given by the [Expansion] axiom. Correct it to $\mathcal{U}'$ so that it has an element $U(p)$ which is unchanged and contains p. Then, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ the old $\mathcal{U}_n$ is subordinated to $\mathcal{U}'$, with the pullback $U_{n,p}$ of $U(p)$ contained in a unique $U'(p) \in \mathcal{U}'$, hence in $U(p) \in \mathcal{U}$. This proves (1).

To prove (2), consider a basis $\mathcal{B}_p$ of our topology on W_0 at p. For each $Y \in \mathcal{B}_p$, take an arbitrary cover $\mathcal{Y}_Y$ of X which includes Y, corrected by Lemma 2.7. Next, by [Expansion], for each $Y \in \mathcal{B}_p$ there is $n = n_Y$ so that $\mathcal{U}_n$ is subordinated to $\mathcal{Y}_Y$ according to Definition 2.5. By uniqueness, $U_{n,p} \subset Y$. So the family $\{U_{n_Y,p} : Y \in \mathcal{B}_p\}$ also constitutes a basis at p, hence (2) holds since our topology is Hausdorff.

(3) Fixed N as in (1), for an arbitrary V choose n_V as above. We find n_V such that $U_V := U_{n_V,p} \subset V$. Then taking adequate pullbacks of the sets in (1) we obtain $\mathrm{Comp}_p\, f^{-n}(U_V) \subset U_V$ for all $n \geq N$.

Finally, writing $n = N + k$ for $n \geq N$ we can write $\mathrm{Comp}_p\, f^{-k}(U_{n_V+N,p}) \subset U_{n_V,p}$ for all $k \geq 0$, yielding (4). □

By taking in the above proof an arbitrary $p \in X$ we prove the following fact, useful e.g. in the Proof of Theorem 2.12.

Lemma 2.8. *If a system (f, W_0, W_1) satisfies the* [Expansion] *axiom with a finite cover $\mathcal{U}$, then the family $\mathcal{U}_n, n = 0, 1, 2, ...$ constitutes a basis of the topology on W_0 at all points of X. Moreover, a basis is constituted by the family $\mathcal{U}_{Mn}$ for every integer $M \geq 1$.*

Proof. As above, for every $p \in X$ and open $V \ni p$ there are $U_{n,p} \in \mathcal{U}_n$ in V for all $n \geq n_V$, in particular for n multiple of M. □

Note that the above Lemma 2.8 remains true if we replace Mn by any sequence $n_i \to \infty$.

Definition 2.9. *A system (f, W_0, W_1) is said to satisfy the* [Irreducibility] *axiom, see* [15]*, if f restricted to its repellor X satisfies the* locally eventually onto (leo) *property, namely for any $x \in X$ and any neighborhood W of x there is some n with $f^n(W) = X$.*

2.3. **Weakly coarse expanding systems.** Let us now give a definition of weakly coarse expanding system.

Definition 2.10. *A system (f, W_0, W_1) is* weakly coarse expanding *if:*

(1) it satisfies the [Expansion] *axiom;*
(2) it satisfies the [Irreducibility] *axiom;*
(3) the branch set B_f is finite;
(4) the repellor X is not a single point.

Note that if $W_0 \subseteq S^2$ is an open subset of the 2-sphere, then the set B_f is always finite (see [32]). Let us note that the last assumption implies that X is uncountable and the topological entropy of f on X is positive.

We shall also use the following lemma ([15, Proposition 2.4.1]), which can be proven using the fact that disjoint compact sets can be separated by disjoint open ones, a consequence of the compactness of $\overline{W_1}$ and the Hausdorff property.

Lemma 2.11. *Let (f, W_0, W_1) be a weakly coarse expanding system with respect to the cover $\mathcal{U}$. Then, there exists N such that for any $U_1, U_2 \in \mathcal{U}_N$ with $U_1 \cap U_2 \neq \emptyset$, there exists $U \in \mathcal{U}$ such that $U_1 \cup U_2 \subseteq U$.*

2.4. **Exponentially contracting metrics.** A very important property of coarse expanding systems is that we can find a metric so that preimages shrink exponentially fast.

Theorem 2.12. *Suppose $f : W_1 \to W_0$ is a finite branched cover and axiom* [Expansion] *holds. Then there exist a metric ρ on X compatible with the topology and constants $C > 0$, $\theta < 1$ such that for all $n \geq 0$*

$$\sup_{U \in \mathcal{U}_n} \mathrm{diam}_\rho(U) \leq C\theta^n.$$

We call a metric ρ which satisfies the above property an *exponentially contracting* metric. Observe that, as the metric ρ is only defined on X, here and later $\operatorname{diam}_\rho(U)$ means the diameter of $U \cap X$ (as in [15]).

An important example (with additional properties) are the *visual metrics* constructed in [15, Chapter 3] and [4, Chapter 8]. Our notion of exponentially contracting metric is more general than the notion of visual metric, as we do not require any lower bound on the diameters of the elements of $\mathcal{U}_n$. Note that by compactness of X any metric ρ on X which induces this topology is complete.

The above theorem is essentially [[15], Theorem 3.2.5]. We will, however, give a complete proof below, using a different method. We use Frink's metrization lemma [13], in the following form.

Lemma 2.13 (Frink's metrization lemma, [24], Lemma 4.6.2)**.** *Let X be a topological space, and let $(\Omega_n)_{n\geq 0}$ be a sequence of open neighborhoods of the diagonal $\Delta \subseteq X \times X$, such that*

(a)
$$\Omega_0 = X \times X$$

(b)
$$\bigcap_{n=0}^{\infty} \Omega_n = \Delta$$
where Δ is the diagonal in $X \times X$.

(c) *For any $n \geq 1$,*
$$\Omega_n \circ \Omega_n \circ \Omega_n \subseteq \Omega_{n-1}$$
where $\circ$ is the composition in the sense of relations: i.e., $R \circ S = \{(x,y) \in X \times X \ : \ \exists z \in X \text{ s.t. } (x,z) \in R \text{ and } (z,y) \in S\}$.

Then there exists a metric ρ on X, compatible with the topology, such that
$$\Omega_n \subseteq \{(x,y) \in X \times X \ : \ \rho(x,y) < 2^{-n}\} \subseteq \Omega_{n-1}$$
for any $n \geq 1$.

Before we prove Theorem 2.12, we prove its instructive metric version, assuming there is an adequate metric on X; this holds if the topology on X has a countable basis [11, Theorem 4.2.8].

Proposition 2.14. *Let $f : W_1 \to W_0$ be a finite branched cover with repellor X, let $\mathcal{U}$ be a finite cover of X by open subsets of W_1 and let ρ be a metric on X such that*
$$\lim_{n\to\infty} \sup\{\operatorname{diam}_\rho(U) \ : \ U \in \mathcal{U}_n\} = 0.$$
Then there exist a metric ρ' on X, which induces the same topology as ρ, and constants $C > 0$, $\theta < 1$ such that
$$\operatorname{diam}_{\rho'}(U) \leq C\theta^n$$
for any $n \geq 0$ and any $U \in \mathcal{U}_n$.

Proof. The proof follows from Lemma 2.13. Let us define V_n as the set of pairs $(x,y) \in X \times X$ such that there exists an element U of $\mathcal{U}_n$ which contains both x and y. Let $\eta > 0$ be the Lebesgue number of $\mathcal{U}_0$ with respect to ρ. By hypothesis there exists an integer $M \geq 1$ such that
$$\sup\{\operatorname{diam}_\rho(U) \ : \ U \in \mathcal{U}_l\} < \eta/3$$
for any $l \geq M$. Now, let us define $\Omega_0 := X \times X$ and $\Omega_n := \bigcup_{k=0}^{M-1} V_{Mn+k}$ for $n \geq 1$. We need to check the hypotheses of Lemma 2.13. Equation (a) is trivially true, as is the inclusion $\Delta \subseteq \bigcap_{n=0}^{\infty} \Omega_n$ in (b). To prove the other inclusion, suppose that $(x,y) \in \Omega_n$ for any $n \geq 0$. Then for any n there exists a connected set $\Gamma_n \in \mathcal{U}_{Mn+k}$, with $0 \leq k \leq M-1$, which contains x, y. Hence
$$\rho(x,y) \leq \sup\{\operatorname{diam}_\rho(U) \ : \ U \in \mathcal{U}_l, l \geq Mn\} \to 0$$
as $n \to \infty$. Thus $x = y$, as claimed.

(Another way is to define $\Omega_n := V_{Mn}$ and refer to Lemma 2.8.)

To prove (c), suppose to have sets $\Gamma_i \in \mathcal{U}_{M+k_i}$ for $i = 1, 2, 3$ with $0 \le k_i < M$ and $\Gamma_i \cap \Gamma_{i+1} \ne \emptyset$ for $i = 1, 2$. Then $\operatorname{diam}_\rho(\Gamma_1 \cup \Gamma_2 \cup \Gamma_3) < \eta$, hence by definition of Lebesgue number for the cover $\mathcal{U}_0 \cap X$ of X, there exists $\Gamma \in \mathcal{U}_0$ such that $(\Gamma_1 \cup \Gamma_2 \cup \Gamma_3) \cap X \subseteq \Gamma$. Hence

$$\Omega_1 \circ \Omega_1 \circ \Omega_1 \subseteq \Omega_0$$

and by taking $f^{M(n-1)}$-preimages,

$$\Omega_n \circ \Omega_n \circ \Omega_n \subseteq \Omega_{n-1}$$

for any $n \ge 1$, proving (c).

Thus, we can apply Lemma 2.13, obtaining that there exists a metric ρ' on X, which induces the same topology as ρ, such that

$$\Omega_n \subseteq \{(x, y) \in X \times X \ : \ \rho'(x, y) < 2^{-n}\} \subseteq \Omega_{n-1} \tag{1}$$

for any $n \ge 1$. Thus for any $U \in \mathcal{U}_n$ we have

$$\operatorname{diam}_{\rho'}(U) < 2^{-\lfloor \frac{n}{M} \rfloor}$$

for any $n \ge 0$, as claimed. Indeed, we have proved that for any $U \in \mathcal{U}_n$ and for any $n \ge 0$ we have $\operatorname{diam}_{\rho'}(U) \le C\theta^n$, with $C = 2 > 0$ and $\theta = 2^{-1/M} < 1$. □

Now let us prove Theorem 2.12 using a similar procedure.

Proof of Theorem 2.12. As in the proof of Proposition 2.14, it is sufficient to check the hypotheses of Frink's lemma.

By Lemma 2.11, the following holds: there exists N such that for all $U_1', U_2' \subset \mathcal{U}_N$, if $U_1' \cap U_2' \ne \emptyset$ then there exists $U \in \mathcal{U}_0$ with $U_1' \cup U_2' \subset U$.

Next, given three sets $U_1', U_2', U_3' \in \mathcal{U}_{2N}$ with $U_i' \cap U_{i+1}' \ne \emptyset$ for $i = 1, 2$, we find $U_1'' \in \mathcal{U}_N$ containing $U_1' \cup U_2'$ and $U_2'' \in \mathcal{U}_N$ containing $U_2' \cup U_3'$ (using the previous fact for two sets and applying f^{-N}). Finally we find $U \in \mathcal{U}_0$ with $U_1'' \cup U_2'' \subseteq U$. Thus we obtain condition (c) of Frink's lemma for $\Omega_0 := X \times X$ and $\Omega_n := \{(x, y) \in X \times X \ : \ \exists U \in \mathcal{U}_{2nN} \text{ s.t. } x, y \in U\}$ for $n \ge 1$. Condition (b) is a consequence of Lemma 2.8.

The rest of the argument follows as in the proof of Proposition 2.14. □

2.5. **Equilibrium states.** Let us now recall some basic notions from ergodic theory. For more details, see e.g. [24].

Let $f : X \to X$ be a continuous map of a compact metric space. A probability measure μ on X is *f-invariant* if $f_\star \mu = \mu$, and we let $M(f)$ be the set of f-invariant probability measures on X. We denote as $h_\mu(f)$ the metric entropy of f with respect to μ.

Now, consider a continuous function $\varphi : X \to \mathbb{R}$, which we call a *potential*. The *topological pressure* of f with potential φ is defined as

$$P_{top}(\varphi) := \sup_{\mu \in M(f)} \left\{ h_\mu(f) + \int_X \varphi \, d\mu \right\}.$$

Note that $P_{top}(\varphi)$ may also be defined topologically ([24], Section 3.2). An f-invariant probability measure μ on X is an *equilibrium state* for φ if it realizes the supremum, namely if

$$P_{top}(\varphi) = h_\mu(f) + \int_X \varphi \, d\mu.$$

In the following, we will consider a weakly expanding system $f : W_1 \to W_0$, and study the equilibrium states on its repellor X.

2.6. **The geometric coding tree.** Let us now construct a symbolic coding for a coarse expanding dynamical system. We start by proving the path lifting property.

Lemma 2.15. *Let $f : Y \to Z$ be a finite branched cover, and let γ be a continuous arc in Z which is disjoint from the set of branch values V_f. Let $x = \gamma(0)$, and $\widetilde{x}$ such that $f(\widetilde{x}) = x$. Then there exists a continuous arc $\widetilde{\gamma}$ in Y such that $f(\widetilde{\gamma}) = \gamma$ and $\widetilde{\gamma}(0) = \widetilde{x}$.*

Proof. Let $p \in \gamma$, and $\widetilde{p}$ a preimage of p. Then since γ is disjoint from V_f, the local degree of f at $\widetilde{p}$ is 1. Hence, there exists a neighborhood U of $\widetilde{p}$ such that $f : U \to f(U)$ is injective, open, and closed, hence a homeomorphism; moreover, $f(U)$ is open. Hence, one can lift $\gamma \cap f(U)$ to an arc $f^{-1}(\gamma) \cap U$ which contains $\widetilde{p}$. Thus, any point p in γ has a neighboorhood U_p over which γ can be lifted, and moreover such that $f^{-1}(U_p)$ is the union of d disjoint open sets which are homeomorphic to U_p; since γ is compact, this implies that the entire γ can be lifted. □

The key point in our approach is that one constructs a semiconjugacy of a weakly coarse expanding system of degree d to the shift map on d symbols.

Let $\Sigma := \{1, \dots, d\}^{\mathbb{N}}$ be the space of infinite sequences of d symbols, and $\sigma : \Sigma \to \Sigma$ the left shift. If $\eta := (\eta_1, \dots, \eta_n) \in \{1, \dots, d\}^n$ is a finite sequence, the *cylinder* associated to η is the set $C(\eta) := \{(\epsilon_i) \in \Sigma \,:\, \epsilon_i = \eta_i \text{ for all } 1 \leq i \leq n\}$. The integer n is called the *depth* of the cylinder $C(\eta)$. Note that for any $n \geq 1$, the set Σ is the disjoint union of d^n cylinders of depth n, and all cylinders are both open and closed. Finally, let us equip Σ with the metric $\rho((\epsilon_i), (\epsilon'_i)) := 2^{-\inf\{k \geq 0 \,:\, \epsilon_k \neq \epsilon'_k\}}$, which we call the *standard metric* (with the convention $2^{-\infty} = 0$).

Proposition 2.16. *Let $f : W_1 \to W_0$ be a weakly coarse expanding system of degree d. Then there exists a Hölder continuous semiconjugacy $\pi : \Sigma \to X$ such that $\pi \circ \sigma = f \circ \pi$. Moreover, if f is locally eventually onto, then π is surjective.*

Proof. The construction is based on the idea of "geometric coding tree" as in [22]. Namely, pick $w \in X \setminus P_f$, which exists since X in uncountable, and let $w_1, \dots, w_d$ be all its preimages. They are in X by backward invariance of X. For each $i = 1, \dots, d$, choose a continuous path γ_i in W_0 connecting w and w_i and avoiding P_f. This exists since the space is strongly path connected and the set P_f is countable.

For each sequence $\alpha = (i_1, i_2, \dots) \in \Sigma$, define $z_n(\alpha)$ by letting $z_0(\alpha) := w_{i_1}$ and $z_n(\alpha)$ for $n \geq 1$ inductively as follows. Let $\gamma_n(\alpha)$ be a curve which is the branch of $f^{-(n-1)}(\gamma_{i_n})$ such that one of its ends is $z_{n-1}(\alpha)$. Such lifts exist by Lemma 2.15, since we chose the curves γ_i to be disjoint from P_f. Then define $z_n(\alpha)$ as the other end of $\gamma_n(\alpha)$.

Now, since X is the intersection of the nested compact sets $K_n := f^{-n}(\overline{W}_1)$, there exists n_0 such that every curve $\gamma_{n_0}(\alpha)$ for $\alpha \in \Sigma$ is contained in the union $\bigcup_{U \in \mathcal{U}_0} U$.

Note that there are d^{n_0} distinct curves of form $\gamma_{n_0}(\alpha)$, parameterized by finite sequences of length n_0: let us denote them $\gamma_{n_0}(\alpha_i)$ with $i = 1, \dots, d^{n_0}$. For each $i = 1, \dots, d^{n_0}$, we write $f_i : [0, 1] \to W_0$ to denote the continuous map whose image is $f_i([0, 1]) = \gamma_{n_0}(\alpha_i) \subset W_0$. Note that the preimage of $\mathcal{U}_0$ under f_i is an open cover of $[0, 1]$. Take a refinement of that cover by open intervals, call it $\mathcal{V}^i$. We then extract a finite subcover $\{V^i_j \in \mathcal{V}^i : 1 \leq j \leq k_i\}$ for some k_i. Set $k = \max\{k_i, i = 1, \dots, d^{n_0}\}$. Now fix n, let $F_i : [0, 1] \to W_0$ be a lift of f_i under f^n, and let $\Gamma_i = F_i([0, 1])$. Note that for each $V \in \mathcal{V}$, its image under any lift F_i as above is a connected subset of $f^{-n}(U)$ for some $U \in \mathcal{U}_0$. Therefore for each $V \in \{V^i_j : 1 \leq j \leq k_i\}$ the image $F_i(V)$ is a subset of some element of $\mathcal{U}_n$. Thus we have at most k open arcs that cover Γ_i, each of them in an element of $\mathcal{U}_n$.

Applying Lemma 2.11 repeatedly and pulling back by f^n, there exists N such that for any $n \geq N$, if $U(1), \dots, U(k)$ is a "chain" of elements of $\mathcal{U}_n$ with $U(j) \cap U(j + 1) \neq \emptyset$ for any $j = 0, \dots, k - 1$, then there exists $U \in \mathcal{U}_{n-N}$ with $U(j) \cup U(j + 1) \subseteq U$.

Now, using the claim proved in the paragraph above, for any $n \geq N$ by the exponential contraction property of Theorem 2.12, we get, summing distances along consecutive pairs in the chain,

$$\rho(z_n(\alpha), z_{n-1}(\alpha)) \leq Ck\theta^{n-N-n_0},$$

and hence

$$\lim_{n\to\infty} z_n(\alpha)$$

exists, and we define $\pi(\alpha)$ as the limit. By construction the map π satisfies $f \circ \pi = \pi \circ \sigma$ and $\pi(\alpha) \in X$. To prove the Hölder continuity, let $\alpha = (i_1, i_2, \dots)$ and $\beta = (j_1, j_2, \dots)$ be two sequences, and let $r := \min\{s \ : \ i_s \neq j_s\}$, so that $z_{r-1}(\alpha) = z_{r-1}(\beta)$ if $r > 1$. Hence

$$\rho(\pi(\alpha), \pi(\beta)) \leq \rho(\pi(\alpha), z_{r-1}(\alpha)) + \rho(z_{r-1}(\beta), \pi(\beta)) \leq 2C'\theta^r$$

with $C' = (kC\theta^{-N-n_0})/(1-\theta)$, which is what we need, as the distance between α and β in the symbolic space is 2^{-r}.

To prove the coding map π is surjective, let $x \in X$. Since f is locally eventually onto, for each $k > 0$ there exists n_k such that $f^{n_k}(B(x, \frac{1}{k})) \supseteq X$. In particular, there exists $w_k \in X$ with $\rho(x, w_k) < \frac{1}{k}$ and such that $f^{n_k}(w_k) = w$. This means, there exists a finite sequence $\alpha_k = (i_1^{(k)}, \dots, i_{n_k}^{(k)})$ such that $w_k = z_{n_k}(\alpha_k)$ and $\lim_{k\to\infty} w_k = x$. By compactness of the shift space, up to passing to a subsequence there exists an infinite sequence $\alpha = (i_1, \dots, i_n, \dots)$ such that $\alpha_k \to \alpha$.

That is, for each m there exists k such that α and α_k coincide for the first m symbols and $n_k \geq m$, hence by the exponential contraction property

$$\rho(z_m(\alpha), z_{n_k}(\alpha_k)) \leq C'\theta^m$$

for some $\theta < 1$. Then

$$\lim_m z_m(\alpha) = \lim_k z_{n_k}(\alpha_k) = x$$

thus $\pi(\alpha) = x$, as claimed. □

3. No entropy drop

Let $f : W_1 \to W_0$ be a finite branched cover with repellor X, let ρ be a metric on X, and let $\varphi : (X, \rho) \to \mathbb{R}$ be a Hölder continuous potential. We now show that if there are no periodic critical points, then the entropy of any equilibrium state on Σ is the same as the entropy of its pushforward on X.

Inspired by [22, Lemma 4], we give the following definition. We say x is an *ϵ-singular point* if $\rho(x, B_f \cap X) < \epsilon$, that is if it lies within distance ϵ of a branch point. Let $(x_i)_{0\leq i\leq n}$ be a finite orbit segment, i.e. a sequence of points such that $f(x_i) = x_{i+1}$. Then we say i is an ϵ-singular time if x_i is an ϵ-singular point.

Lemma 3.1. *Suppose that no critical point is periodic. Then for any $0 < \zeta < 1$, there exists $\epsilon > 0$ such that for any $x \in X$ and any finite orbit segment $(x_i)_{0\leq i\leq n}$ with $x_n = x$ we have*

$$\#\{0 \leq i \leq n \ : \ i \text{ is an } \epsilon\text{-singular time}\} \leq \zeta n + p,$$

where p is the number of critical points.

Proof. Let p be the number of critical points. Given $\zeta < 1$, let us choose k such that $\frac{p}{k} < \zeta$. Fix a critical point c, and let $\beta := \rho(c, \{f^n(c)\}_{1\leq n\leq k}) > 0$. We denote as $B(x, r)$ the open ball of radius r and center x for the metric ρ.

We readily see that there exists $\epsilon > 0$ such that if $y \in B(c, \epsilon)$, then $f^l(y) \notin B(c, \epsilon)$ for any $l \leq k$. Indeed, by continuity of f, there exists $0 < \epsilon < \beta/2$ such that $f^l(B(c, \epsilon)) \subseteq B(f^l(c), \beta/2)$ for any $l \leq k$. Then if $y \in B(c, \epsilon)$ then $f^l(y) \in B(f^l(c), \beta/2)$, which is disjoint from $B(c, \epsilon)$ since $\epsilon < \beta/2$.

Moreover, we further shrink ϵ so that if $x \in X$ and the ball $B(x, \epsilon)$ does not intersect the branch set B_f, then the restriction $f|_{B(x,\epsilon)}$ is injective.

Now suppose (x_i) is an orbit segment, and $i < j$ are two singular times such that x_i and x_j lie within distance η of the same critical point c. Then by the argument above $j - i \geq k$. Thus, the number of singular times is at most $p\lceil \frac{n+1}{k} \rceil \leq p(\frac{n}{k} + 1) < \zeta n + p$, as desired. □

Let us denote as $\sigma : \Sigma \to \Sigma$ the shift map on the symbolic space and $f : W_1 \to W_0$ the topological branched cover, with repellor X. Moreover, let $\pi : \Sigma \to X$ denote the semiconjugacy from Proposition 2.16.

Lemma 3.2. *Suppose that no critical point is periodic. For any $x \in X$, denote as $S_{n,x}$ the number of cylinders of depth n which intersect $\pi^{-1}(x)$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \log \sup_{x\in X} S_{n,x} = 0.$$

Proof. We shall use the notation $z_n(\alpha)$ for the vertices of the geometric coding tree as in the proof of Proposition 2.16. First, by exponential convergence of backward orbits, there exist $C > 0$, $\lambda < 1$ such that

$$\rho(z_n(\alpha), \pi(\alpha)) \leq C\lambda^n$$

for any $\alpha \in \Sigma$ and any $n \geq 0$.

Fix ζ with $0 < \zeta < 1$, and let $\epsilon > 0$ such that Lemma 3.1 is satisfied for ζ. Then there exists N such that $C\lambda^m \leq \epsilon$ for any $m \geq N$.

Let $G := \{z_n(\beta) \ : \ \beta \in \pi^{-1}(x)\}$. Consider the subtree of the geometric coding tree formed by the union of the paths which connect each element of G with the root w. We say an integer $0 \leq k \leq n-1$ is a *branching level* for G if there are two elements $z \neq z'$ in G with $f^{k+1}(z) = f^{k+1}(z')$ but $f^k(z) \neq f^k(z')$. Suppose that $k \leq n - N$ is a branching level for G, and let $z = z_n(\beta)$ and $z' = z_n(\beta')$ be two distinct points in G with $f^{k+1}(z) = f^{k+1}(z')$ but $f^k(z) \neq f^k(z')$. Note that by definition we have $f^k(x) = \pi(\sigma^k\beta)$ as well as $f^k(z) = z_{n-k}(\sigma^k\beta)$.

Now, by the above exponential convergence, we have

$$\rho(f^k(z), f^k(x)) = \rho(z_{n-k}(\sigma^k\beta)), \pi(\sigma^k\beta)) \leq C\lambda^{n-k} < \epsilon$$

and similarly

$$\rho(f^k(z'), f^k(x)) \leq C\lambda^{n-k} < \epsilon$$

hence f is not injective on the ball of radius ϵ around $f^k(x)$, thus k is an ϵ-singular time for x.

Thus, a branching level for G either satisfies $n - N \leq k \leq n - 1$ or is an ϵ-singular time for x. By Lemma 3.1 the number of ϵ-singular times in the forward orbit $(x, f(x), \dots, f^n(x))$ is at most $\zeta n + p$, thus the number of branching levels in G is at most $N + \zeta n + p$. Hence, the number of elements in G is bounded above by

$$S_{n,x} = \#G \leq d^{N+\zeta n+p}$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \log \sup_{x\in X} S_{n,x} \leq \zeta \log d.$$

By taking $\zeta \to 0$ we obtain that $\limsup \leq 0$. Further, since $S_{n,x} \geq 1$ the $\liminf$ is ≥ 0, which implies the claim.

□

Lemma 3.3. *Let μ be a σ-invariant measure on the symbolic space Σ, and let $\nu = \pi_\star\mu$ be the pushforward measure on X. Then*

$$h_\mu(\sigma) = h_\nu(f).$$

Proof. Let $\mathcal{A}^n$ denote the partition of Σ into cylinders of depth n, let θ be the partition in preimages of points under π, and for ν-a.e. $x \in X$ let μ_x be the conditional measure on the fiber over x. Let us consider the *relative entropy* $h_\mu(\sigma|f)$ (see e.g. [31]), which we recall satisfies

$$h_\mu(\sigma) = h_\nu(f) + h_\mu(\sigma|f).$$

By definition of relative entropy,

$$\begin{aligned} h_\mu(\sigma|f) &= \lim_{n\to\infty} \frac{1}{n} H_\mu(\mathcal{A}^n|\theta) \\ &= \lim_{n\to\infty} \frac{1}{n} \int_X d\nu(x) \sum_{a\in\mathcal{A}^n} -\mu_x(a)\log\mu_x(a) \end{aligned}$$

and, by the comparison between measure-theoretic and topological entropy,

$$\leq \limsup_{n\to\infty} \frac{1}{n} \int_X \log S_{n,x}\, d\nu(x) = 0,$$

where the last claim follows from Lemma 3.2. □

4. Existence and uniqueness of equilibrium states

We start by showing that one can "lift" invariant measures.

Lemma 4.1. *Let $\sigma : \Sigma \to \Sigma$ and $\tau : X \to X$ be continuous maps, and let $\pi : \Sigma \to X$ be a continuous semiconjugacy, i.e. so that $\pi \circ \sigma = \tau \circ \pi$. Then for any τ-invariant probability measure μ on X, there exists a probability measure $\widetilde{\mu}$ on Σ which is σ-invariant and such that $\pi_\star\widetilde{\mu} = \mu$.*

Proof. We will use Riesz' extension theorem. Namely, the measure μ defines a positive functional on $C_0(\Sigma) := \{f \in C(\Sigma) \ : \ \exists g \in C(X) \text{ with } f = g \circ \pi\}$ by setting

$$\mu(g\circ\pi) := \int g\, d\mu.$$

The functional is well-defined since π is surjective. Now, we need to check that

$$C(\Sigma) = C_0(\Sigma) + P$$

where P is the convex cone of non-negative functions. This is obvious since given $f \in C(\Sigma)$ we have

$$f = \inf f + (f - \inf f)$$

where $\inf f$ is constant, hence belongs to $C_0(\Sigma)$, and $f - \inf f \geq 0$. Then by the Riesz' extension theorem ([27, Theorem 5.5.8]) there exists a positive functional on $C(\Sigma)$ which extends μ. By Riesz' representation theorem, this functional is represented by a measure ν on Σ which has the property that

$$\int f\circ\pi\, d\nu = \int f\, d\mu$$

for any $f \in C(X)$. Now, by taking a limit point of the sequence

$$\mu_n := \frac{1}{n}\sum_{j=0}^{n-1} \sigma_\star^j \nu$$

we obtain a measure $\widetilde{\mu}$ on Σ which is σ-invariant and satisfies $\pi_\star\widetilde{\mu} = \mu$. □

Then, we show that the pushforward of an equilibrium state on Σ is an equilibrium state on X.

Lemma 4.2. *Let $\varphi : (X,\rho) \to \mathbb{R}$ be a Hölder continuous potential, and let $\widetilde{\mu}$ be an equilibrium state for $\widetilde{\varphi} = \varphi\circ\pi$. Then $\mu := \pi_\star\widetilde{\mu}$ is an equilibrium state for φ.*

Proof. Since π is a semiconjugacy,

$$P_{top}(\varphi) \leq P_{top}(\varphi \circ \pi)$$

and since $\widetilde{\mu}$ is an equilibrium state

$$= h_{\widetilde{\mu}}(\sigma) + \int \widetilde{\varphi}\ d\widetilde{\mu}$$

and using Lemma 3.3 (no entropy drop)

$$= h_{\mu}(f) + \int \varphi\ d\mu \leq P_{top}(\varphi)$$

where in the last step we used the variational principle. Hence

$$P_{top}(\varphi) = P_{top}(\varphi \circ \pi)$$

and μ is an equilibrium state for φ. □

Lemma 4.3. *Let μ be an equilibrium state for φ. Then there exists a measure $\widetilde{\mu}$ which is an equilibrium state for $\widetilde{\varphi}$ and $\pi_\star\widetilde{\mu} = \mu$.*

Proof. By the proof of Lemma 4.2,

$$P_{top}(\varphi \circ \pi) = P_{top}(\varphi)$$

and, since μ is an equilibrium state,

$$= h_{\mu}(f) + \int \varphi\ d\mu$$

and, using that entropy increases by taking an extension and that $\pi_\star\widetilde{\mu} = \mu$,

$$\leq h_{\widetilde{\mu}}(\sigma) + \int \widetilde{\varphi}\ d\widetilde{\mu}$$

hence by the variational principle $\widetilde{\mu}$ is an equilibrium state. □

Lemma 4.4. *For any Hölder continuous potential $\varphi : X \to \mathbb{R}$ there is a unique equilibrium state.*

Proof. Let μ_1, μ_2 be two equilibrium states for φ on X. Then the measures $\widetilde{\mu_i}$ for $i = 1, 2$ produced in the previous Lemma are equilibrium states for $\varphi \circ \pi$ on Σ, and we know that this is unique. Hence $\mu_1 = \pi_\star\widetilde{\mu_1} = \pi_\star\widetilde{\mu_2} = \mu_2$. □

Proof of Theorem 1.1 (1)-(5). Consider a Hölder continuous potential $\varphi : X \to \mathbb{R}$ and a Hölder continuous observable $\psi : X \to \mathbb{R}$, and let $\pi : \Sigma \to X$ be the semiconjugacy of Proposition 2.16. Then $\varphi \circ \pi$ and $\psi \circ \pi$ are Hölder continuous with respect to the metric on Σ. Now, by the previous discussion there is a unique equilibrium state μ_φ on X for the potential φ and a unique equilibrium state $\nu_{\varphi\circ\pi}$ for the potential $\varphi \circ \pi$. Moreover, it is well-known that equilibrium states for Hölder potentials on the full shift space on d symbols satisfy the statistical laws (CLT, LIL, EDC, LD); for CLT and LIL, see [24, Theorem 5.7.1]; for EDC, see [24, Theorem 5.4.9], and for LD see for example [10] and references therein. Hence the sequence $(\psi \circ \pi \circ \sigma^n)_{n\in\mathbb{N}}$ satisfies the statistical laws with respect to $\nu_{\varphi\circ\pi}$. Since $\psi \circ \pi \circ \sigma^n = \psi \circ f^n \circ \pi$ and $\nu_{\varphi\circ\pi}$ pushes forward to μ_φ, the sequence $(\psi \circ f^n)_{n\in\mathbb{N}}$ also satisfies CLT, LIL, EDC and LD with respect to μ_φ. □

5. Periodic critical points

Let us now focus on the case where $f : W_1 \to W_0 \subseteq S^2$ is defined on an open subset of the 2-sphere, but periodic critical points are allowed. In this case, we can blow up the sphere along the preimages of the critical orbits and obtain a coarse expanding map of a Sierpiński carpet (or a subset thereof) without periodic critical points. Our construction turns out to be the inverse of the construction used in [14].

The main result of this section is the following proposition, whose proof we will give in several steps.

Proposition 5.1. *Let $f : W_1 \to W_0 \subseteq S^2$ be a weakly coarse expanding map on an open subset of the 2-sphere, and let ρ be an exponentially contracting metric on X. Then there exist a strongly path connected space $\widetilde{W_0}$ and a weakly coarse expanding system $g : \widetilde{W_1} \to \widetilde{W_0}$ without periodic critical points, with repellor Y, and a metric ρ' on Y which is exponentially contracting with respect to g, and there is a continuous map $\pi : \widetilde{W_0} \to W_0$ such that $\pi \circ g = f \circ \pi$.*

Let us start the proof of Proposition 5.1 by obtaining a local model in a neighborhood of a fixed critical point.

Lemma 5.2. *Let $f : W_1 \to W_0 \subseteq S^2$ be a weakly coarse expanding map, and let $p \in X$ be a fixed critical point. Then for any $\lambda > 1$ there exist $d \in \mathbb{Z} \setminus \{0\}$, a neighborhood U of p and a homeomorphism $h : \overline{\mathbb{D}} \to \overline{U}$ such that $f \circ h = h \circ g$ where $g : \overline{\mathbb{D}}_{\lambda^{-1}} \to \overline{\mathbb{D}}$ is defined as*

$$g(re^{i\theta}) := \lambda re^{id\theta}$$

for any $r \leq 1$, $\theta \in \mathbb{R}$.

Remark 1. Note that d may be negative if f is orientation reversing in a neighborhood of p. The absolute value $|d|$ is the local degree of f at p.

Proof. Let B be an open topological disc in S^2 containing the branched f-fixed point p and whose closure does not contain other critical points, small enough so that all connected components of $f^{-1}(B)$ which do not contain p are disjoint from B.

Claim. There exists a Jordan curve γ in $B \setminus \{p\}$ such that a component of $f^{-1}(\gamma)$ is a Jordan curve disjoint from γ and separates γ from p.

Proof of the Claim. By [Expansion], using Lemma 2.6, there exist $N > 0$ and an open set U, with $p \in U \subseteq B$, such that for $n \geq N$, the closure of $U_n := \mathrm{Comp}_p f^{-n}(U)$ is contained in U. If we replace U by U_N, its pullbacks $U_k, k = 0, 1, \dots$ are all in B, so their closures do not contain critical points except possibly p.

Replace finally U_N by a smaller *Jordan domain* $\hat{U}$ containing p, that is an open topological disc with its boundary being a Jordan curve. Let $V := \mathrm{Comp}_p \left(\bigcap_{n=0}^{N-1} \hat{U}_n \right)$, where Comp_p means the component containing p. The pullbacks $\hat{U}_n$ of the Jordan domain $\hat{U}$ must be Jordan domains, since they do not contain other critical points, see [32]. Since every connected component of the intersection of finitely many Jordan domains in the plane is a Jordan domain (see e.g. [9, Proposition 2.4]), then V is a Jordan domain, that is $\gamma := \partial V$ is a Jordan curve.

Similarly $V_1 := \mathrm{Comp}_p \left(\bigcap_{n=0}^{N-1} \hat{U}_{n+1} \right)$ is also a Jordan domain, and V_1 is a pullback of V, since $f^{-1}(\hat{U}_{N-1})$ has only one component in B hence we can change the order of f^{-1} and Comp_p. Moreover $V_1 \subseteq V$. Let $\gamma_1 := \partial V_1$, which is also a Jordan curve. It is a pullback of γ.

Now, if the boundaries of V and V_1 are disjoint, the claim is satisfied. Otherwise, we shall correct V so that the closure of V_1 is contained in V. To this end, denote $a := \gamma \cap \gamma_1$. If $f(a) \subset a$ then a is non-escaping, contradicting the [Expansion] axiom (in particular, Lemma 2.6).

Otherwise, we modify γ as follows. Note that a and $f(a)$ are closed sets, $f(a) \subset \gamma$, and $\gamma \setminus a$ is the union of countably many open arcs, hence $f(a) \setminus a$ is a (non-empty) subset of $\gamma \setminus a$. Now, we consider a neighborhood of $f(a) \setminus a$ in $\gamma \setminus a$, and modify there a part of γ by pushing it slightly into V, so that it is still disjoint from γ_1; note that we do not move points in the closure of $f(a) \setminus a$ belonging to a. This way, we obtain a new Jordan curve γ^1 which satisfies $\gamma^1 \cap f^{-1}(\gamma^1) = a \cap f^{-1}(a)$.

If the latter intersection is not empty, we repeat this process by considering for any n the set

$$a_n := \{x \in a \ : \ f^k(x) \in a \text{ for all } k = 1, \ldots, n\},$$

and we modify in $\gamma^{n-1} \setminus a_{n-1}$ a neighborhood of $f(a_{n-1}) \setminus a_{n-1}$ to obtain a new curve γ^n so that $\gamma^n \cap f^{-1}(\gamma^n) = a_n$.

Now, we claim there exists N such that $a_N = \emptyset$. Indeed, otherwise the set $a_\infty = \{x \in \gamma : f^n(x) \in a \ \forall n \geq 1\}$ would be non-empty with $f(a_\infty) \subset a_\infty$, contradicting the [Expansion] axiom.

Thus, the inductive procedure stops after N steps, and γ^N is disjoint from the preimage $f^{-1}(\gamma^N)$, completing the proof of the claim.

□

Let us now complete the proof of the lemma. By induction, let us define as γ_0 the curve γ given by the previous claim, and as $\gamma_{n+1} := f^{-1}(\gamma_n) \cap U$, obtaining a sequence of disjoint, nested simple closed curves which contain p in their interior. By the [Expansion] property, the diameter of γ_n converges to 0. For each n, let us define as R_n the annulus bounded by γ_n and γ_{n+1}. Let us pick a point p_0 on γ_0, and let us choose as $p_1 \in \gamma_1$ as one of the d preimages of p_0. Let us pick a continuous arc $\alpha = \alpha_0$ which joins p_0 and p_1 and is contained in R_0. Then for each $n \geq 1$ let us define as α_{n+1} the component of $f^{-1}(\alpha_n)$ which starts at p_n, and let p_{n+1} be other end of α_{n+1}. Let $\beta := \bigcup_{n=0}^\infty \alpha_n$, which is a continuous (open) arc joining p_0 to p. Note that by construction, $f(\beta) \cap U \subseteq \beta$. Now, for each n denote as $\beta_{n,j}$ for $0 \leq j \leq d^n - 1$ the connected components of $f^{-n}(\beta) \cap U$.

Let us now fix $\lambda > 1$, and consider the map $g : \mathbb{D} \to \mathbb{D}$ defined as $g(re^{i\theta}) := \lambda r e^{id\theta}$. Let $\gamma_0' = \partial\mathbb{D}$ and for each n denote as $\gamma_n' := g^{-n}(\gamma_0')$. Moreover, let $\beta' := (0, 1] \subseteq \mathbb{D}$, and for each n let $\beta_{n,j}'$ for $0 \leq j \leq d^n - 1$ denote the components of $g^{-n}(\beta')$. Finally, let us define the annuli $R_n' \subseteq \mathbb{D}$ bounded by the curves γ_n'.

Now let us construct h. Define h on γ_0' as an arbitrary orientation-preserving homeomorphism $h : \gamma_0' \to \gamma_0$, such that $h(1) = p_0$. Next, lift it to $h : \gamma_1' \to \gamma_1$ and proceed inductively lifting it to $h : \gamma_n' \to \gamma_n$ for any $n \in \mathbb{N}$, so that $f \circ h = h \circ g$ and $h(\lambda^{-n}) = p_n$. This is possible because both g and f have the same degree $d \in \mathbb{Z} \setminus \{0\}$ (see [29, Corollary 2.5.3]).

Then, extend h continuously to a homeomorphism $h : \alpha' \to \alpha$, where $\alpha' = \alpha_0' := [\lambda^{-1}, 1] \subset \overline{\mathbb{D}}$. Next, extend h to a homeomorphism between the closed sets R_0' and R_0 using as charts the Riemann mappings $\Phi' : \mathbb{D} \to R_0' \setminus \alpha_0'$ and $\Phi : \mathbb{D} \to R_0 \setminus \alpha_0$ and their extensions to homeomorphisms (local, because two arcs are glued together to α) $\hat{\Phi}'$ and $\hat{\Phi}$ on $\overline{\mathbb{D}}$, which exist by Carathéodory's theorem. Extend $\hat{\Phi}^{-1} \circ h \circ \hat{\Phi}'$ from $\partial\mathbb{D}$ to h' on $\overline{\mathbb{D}}$ radially and next define the extended h by $\hat{\Phi} \circ h' \circ (\hat{\Phi}')^{-1}$.

Finally, define inductively $h : R_n' \to R_n$ for $n = 1, 2, \ldots$ as lifts of $h : R_0' \to R_0$, extending continuously the map h already defined on $\partial R_n'$. We complete the construction by setting $h(0) := p$.

Note that by this construction for any n, j we have $h(\beta_{n,j}') = \beta_{n,j}$ (if for each n they are indexed by j's in the same, say "geometrical", order). Therefore, h maps the "rectangles" bounded by $\gamma_n', \gamma_{n+1}', \beta_{n,j}', \beta_{n,j+1}'$ to the "rectangles" bounded by $\gamma_n, \gamma_{n+1}, \beta_{n,j}, \beta_{n,j+1}$. □

Proof of Proposition 5.1. By Lemma 5.2, we can topologically conjugate the map f on a neighborhood of each periodic critical point to a map which linearly expands radial distances. Then using the linear model we can blow up each point in the periodic critical orbit to a circle, and define the map on each circle as multiplication by the local degree. Moreover, we also need to blow up points in the backward orbit of the critical point and then we obtain a map $g : \widetilde{W}_1 \to \widetilde{W}_0 \subseteq \widetilde{S}$, where $\widetilde{S}$ is the Sierpiński carpet obtained by replacing any point in the backward orbit of a critical point by a circle. This space is still strongly path connected, so we can apply the techniques of the previous section.

Construction of the blowup. In order to discuss the details of this construction, let $\mathcal{C}$ denote the (finite) set of periodic critical points which lie in X, and let $\mathcal{E} = \bigcup_{n\geq 0} f^{-n}(\mathcal{C})$. We can define a space $\widetilde{S}$ which is given by blowing up every point of $\mathcal{E}$ to a circle. Namely, for each $q \in \mathcal{E}$ let S_q be a copy of S^1. The space $\widetilde{S}$ is defined as a *set* as

$$\widetilde{S} := (S^2 \setminus \mathcal{E}) \sqcup \bigsqcup_{q\in\mathcal{E}} S_q.$$

The topology on $\widetilde{S}$ will be defined shortly. Note there is a natural projection map $\pi : \widetilde{S} \to S^2$ which sends each S_q to q. Let $\widetilde{W_0} := \pi^{-1}(W_0)$, $\widetilde{W_1} := \pi^{-1}(W_1)$.

Definition of g. Let us now extend f to a map $g : \widetilde{W_1} \to \widetilde{W_0}$. In order to do so, let us identify all S_q for $q \in \mathcal{E}$ with $\mathbb{R}/\mathbb{Z}$, and define $g : S_q \to S_{f(q)}$ as $g(\theta) := d\theta \mod 1$, where d is the local (signed) degree of f at q. Finally, let us define $g := f$ on $\widetilde{W_1} \setminus \bigcup_{q\in\mathcal{E}} S_q$.

Definition of the topology. Now, let us define a topology on $\widetilde{W_0}$ as follows.

Let us choose one element from any periodic orbit in X containing critical points, and let us call $\mathcal{E}^*$ the union of such points. Moreover, for each $q \in \mathcal{E}$ let $m = m(q)$ be the minimal $m \geq 1$ such that $f^m(q) \in \mathcal{E}^*$. In particular for $p \in \mathcal{E}^*$ this is the minimal period.

Let now $p \in \mathcal{E}^*$. Then by Lemma 5.2, there exists a neighborhood U_p of p such that the map $f^m : U_p \to f^m(U_p)$ is topologically conjugate to $(r, \theta) \mapsto (\lambda r, d\theta \mod 1)$ for some $\lambda > 1$ and $|d| = \deg(f^m; p)$ is the local degree of f^m at p.

We now fix some small values $r_0, \epsilon > 0$ and define for any $\theta_0 \in [0, 2\pi)$ the set

$$V_{p,\theta_0} := \pi^{-1}(h(\{0 < r < r_0, \theta_0 - \epsilon < \theta < \theta_0 + \epsilon\})) \cup \{\theta \in S_p \ : \ \theta_0 - \epsilon < \theta < \theta_0 + \epsilon\}.$$

Suppose now $q \in \mathcal{E} \setminus \mathcal{E}^*$, let $m = m(q)$ and $p = f^m(q)$. By [33, Theorem X.5.1], there exist a neighborhood U_q of q and a homeomorphism $h' : \mathbb{D} \to U_q$ such that $h^{-1} \circ f^m \circ h'(re^{i\theta}) = \lambda re^{id\theta}$, where d is the local degree of f^m at q and $\lambda > 0$. Similarly as above, we define for any $\theta_0 \in [0, 2\pi)$

$$V_{q,\theta_0} := \pi^{-1}(h'(\{0 < r < r_0, \theta_0 - \epsilon < \theta < \theta_0 + \epsilon\})) \cup \{\theta \in S_q \ : \ \theta_0 - \epsilon < \theta < \theta_0 + \epsilon\}.$$

Remark 2. Note that in the above construction the change of coordinates in the domain and range are different; thus, we do not claim the map to be topologically conjugate to its local model, as we do in Lemma 5.2. Moreover, here, just to define a basis of our topology at S_q it is sufficient to consider the chart $h' = h_q$ for each q separately. In fact, similarly one could define these charts inductively as a *compatible system*, in the sense that $f \circ h_q = h_{f(q)} \circ F_q$ with e.g. $F_q(re^{i\theta}) := \lambda^{1/m(q)} re^{id\theta}$, where $d = \deg(f, q)$. As a consequence of Lemma 5.2, this would also hold for $f^{m(p)}$ and $h_p = h_{f^{m(p)}} = h$ with $p \in \mathcal{E}^*$.

We now define the topology on $\widetilde{W_0}$ to be the topology generated by

$$\{\pi^{-1}(U) \ : \ U \text{ open in } W_0\} \cup \{g^{-n}(V_{p,\theta_0}) \ : \ n \geq 0, p \in \mathcal{C}, \theta_0 \in [0, 2\pi)\}.$$

Note that by the above construction it is immediate that the projection $\pi : \widetilde{W_0} \to W_0$ is continuous. Moreover, let us check that g is also continuous with respect to this topology. First, if $\widetilde{U} = \pi^{-1}(U)$ with U open in W_0, then $g^{-1}(\widetilde{U}) = \pi^{-1}(f^{-1}(U))$ is open in $\widetilde{W_0}$ since f and π are continuous. If, on the other hand, $\widetilde{U} = g^{-n}(V_{p,\theta_0})$ for some n, p, θ_0, then $g^{-1}(\widetilde{U}) = g^{-n-1}(V_{p,\theta_0})$ is also open by construction, hence g is continuous.

The topology is Hausdorff since for each $x, y \in \widetilde{W_0}$ such that $\pi(x) \neq \pi(y)$, there exist open disjoint sets U_1, U_2 in W_0 which separate $\pi(x)$ and $\pi(y)$, as the topology on W_0 is Hausdorff; then their preimages, which are open by our definition of topology on $\widetilde{W}_0$, separate x and y. The only case to be checked is when x, y belong to one S_q, but then they are separated by finite intersections of the sets $g^{-n}(V_{g^n(q),\theta_0})$.

Finally, note that $\widetilde{W}_0$ is metrizable, since it is T_3 and has a countable basis ([11, Theorem 4.2.9]). Moreover it can be homeomorphically embedded in S^2, see Appendix B.

Compactness. We now show that the closure of $\widetilde{W}_1$ is compact.

Given a family $\mathcal{W}$ of open sets covering $\widetilde{S}$, we can assume they belong to a basis of the topology. Those of the form $\pi^{-1}(U)$ for $U \in \mathcal{U}$, the family of all open sets in S^2, will be called of type I, those intersecting S_q of type II$_q$.

Given $q \in \mathcal{E}$, by the compactness of S_q we can choose a finite family $\mathcal{W}_q$ of sets of the cover of type II$_q$ which cover S_q. Their union contains an open neighborhood V_q of S_q, of type I (since $\mathcal{U}$ separates points). Pick an open set V'_q of type I so that $V_q \cup V'_q = \widetilde{S}$, and replace the cover $\mathcal{W}$ by a finer one, by intersecting all its elements with V_q and V'_q.

Do it for all $q \in \mathcal{E}$ getting a resulting cover $\mathcal{W}'$. Consider the cover consisting of all the sets of type I in $\mathcal{W}$ and of the sets arising by replacing each finite family $\mathcal{W}_q$ by one set V_q: this is a cover of $\widetilde{S}$ by sets of type I, so we can find a finite subcover due to the compactness of S^2.

Replace back all V_q in this finite cover by the finite family $\mathcal{W}_q$. This gives a finite cover of $\widetilde{S}$ refining the original cover $\mathcal{W}$. Compactness is proved.

Local connectivity. To show local connectivity of $\widetilde{W}_0$, we show that every point $x \in \widetilde{W}_0$ has a basis of (strongly) path connected neighborhoods. As usual, there are two cases.

Case 1. Suppose $x \notin \pi^{-1}(\mathcal{E})$. Consider an open connected set $U \subseteq W_0 \subseteq S^2$ which contains $\pi(x)$, with the Euclidean topology. Then, since $\mathcal{E}$ is countable and U is a connected subset of the sphere, the set $U \setminus \mathcal{E}$ is (strongly) path connected. Now, by construction the projection map π is a homeomorphism between the set $\pi^{-1}(U) \setminus \bigcup_{q\in\mathcal{E}} S_q$ and $U \setminus \mathcal{E}$. Thus, every point of $\widetilde{W}_0$ which is not in $\pi^{-1}(\mathcal{E})$ has a neighborhood basis which is strongly path connected.

Case 2. If x belongs to some S_p for some periodic critical point p, then consider r_0 as before, and for any $\epsilon > 0, \theta_0 \in [0, 2\pi)$ the set

$$\mathcal{V} := \pi^{-1}(h(\{0 < r < r_0, |\theta - \theta_0| < \epsilon\})) \cup \{\theta \in S_p \ : \ |\theta - \theta_0| < \epsilon\}.$$

Then, the set $\mathcal{V} \setminus \bigcup_{q\in\mathcal{E}\setminus\{p\}} S_q$ is homeomorphic to

$$\{(r, \theta) \in \mathbb{R}^2 \ : \ 0 \leq r < r_0, \theta_0 - \epsilon < \theta < \theta_0 + \epsilon\} \setminus E$$

where E is a countable set, hence it is also strongly path connected.

Finally, if $x \in S_q$ for some $q \in f^{-n}(\mathcal{C})$, then one repeats the previous argument considering the connected components $\mathcal{V}'$ of $g^{-n}(\mathcal{V})$, where $\mathcal{V}$ is defined as above.

The repellor Y and the [Irreducibility] axiom. Now, we denote as Y the repellor for $g : \widetilde{W}_1 \to \widetilde{W}_0$.

Note that by construction $Y = \pi^{-1}(X)$, so it contains S_p for any $p \in \mathcal{E}$. Suppose for simplicity that p is a fixed point for f. Note now that S_p is contained in the closure of the lift of $X \setminus \{p\}$. To prove this, note that X is an uncountable perfect set, so there is $x \in X$ arbitrarily close to p, and consider the set of all its f^n-preimages in $U = h(\mathbb{D})$ (see Lemma 5.2). Since the repellor X is backward invariant, these preimages belong to X, hence their lifts (points and circles) belong to Y since $Y = \pi^{-1}(X)$. But due to the degree of f at p being $d \geq 2$, they have arguments in the coordinates h being all numbers of the form $(\theta_n + 2\pi j)/d^n$ for a constant $0 \leq \theta_n < 2\pi$ and integers $j : 0 \leq j < d^n$, accumulating on the whole S_p as $n \to \infty$.

In particular we obtain the locally eventually onto (leo) property of $g|_Y$. Indeed, if W is an open neighbourhood of $z \in S_p$ in Y for p a fixed point, then it contains a point $\tilde{x}$ whose projection by

π is $x \notin S_p$ as above. Therefore there exists W' a neighbourhood of $\tilde{x}$ in Y being the lift of an open set $\pi(W')$ in X. By the leo property of f on X there is n such that $f^n(\pi(W')) = X$. Hence $g^n(W) \supset g^n(W') = Y$. For q in the grand orbit of a periodic critical point p we use $g = f^{m(q)}$ and the definition of the basis of the topology at points in S_q as the pullbacks of the basis at S_p. For W in the basis of the topology at the points z not in any S_q we rely on the fact that, by construction, the basis at z is the lift of the basis downstairs, at $\pi(z)$.

A contracting metric. We claim that $(g, \widetilde{W}_0, \widetilde{W}_1)$ with an appropriate cover of Y is a weakly coarse expanding system without periodic critical points.

Let us assume for simplicity (see also Remark 4) that there is only one periodic critical point p in X with $f(p) = p$, and let d be the local degree of f at p.

Let $X \subset W_1$ denote our f-invariant compact repellor. Let ρ_1 be a metric on X compatible with the spherical topology given by the metric ρ_e^X defined as the restriction to X of the spherical metric ρ_e on S^2. Due to compactness both metrics are equivalent, that is for every $\epsilon > 0$ there exists δ such that $\rho_1(x, y) < \delta$ implies $\rho_e^X(x, y) < \epsilon$, and vice versa.

Given a cover $\mathcal{U}$ and a metric ρ, we use the notation $\mathrm{mesh}_\rho(\mathcal{U}) := \sup_{U \in \mathcal{U}} \mathrm{diam}_\rho(U)$. Let $\mathcal{U}_0$ be our cover of a neighborhood of X and $\mathcal{U}_n$ the pullback covers of a neighborhood of X for f^{-n}. By [Expansion] we have

$$\mathrm{mesh}_{\rho_1}(\mathcal{U}_n \cap X) \to 0$$

as $n \to \infty$, hence the same holds for ρ_e^X. We measure here the elements of $\mathcal{U}_n$ intersected with X. Moreover, as $n \to \infty$

$$\mathrm{mesh}_e(\mathcal{U}_n) \to 0 \tag{2}$$

in the spherical metric ρ_e in S^2 without intersecting with X, by Definition 2.5.

The Euclidean distance in $\overline{\mathbb{D}}$ being the domain of the chart h in Lemma 5.2 will be denoted by $\rho_{e'}$. The same symbol will be used for the image of the metric on the range set U. The metrics $\rho_{e'}$ and ρ_e are equivalent on U by compactness.

By replacing $\mathcal{U}_0$ with some $\mathcal{U}_n$ (to be our new $\mathcal{U}_0$), assume

$$\mathrm{diam}_{e'}(U_n \cap h(\mathbb{D})) \leq \frac{1}{3}(1 - \lambda^{-2}) \tag{3}$$

for any $n \geq 0$ and any $U_n \in \mathcal{U}_n$, where $\lambda > 1$ is given by Lemma 5.2.

We can assume that $U_0(p) = h(B(0, \lambda^{-N_0}))$, for an arbitrary N_0, is the set in $\mathcal{U}_0$ covering our branching fixed point p. This may be realized by first replacing $\mathcal{U}_0$ by $\mathcal{U}_n$ for an appropriate n so that $U_n \in \mathcal{U}_n$, which covers p, is small enough; and next, replacing that U_n by its superset $h(B(0, \lambda^{-N_0}))$.

For any connected $Z \subset h(\overline{\mathbb{D}} \setminus \{0\})$ denote by $\Delta(Z)$ the oscillation of the angle θ on Z. Choose $\hat{r} > 0$ such that if $A \subseteq h(\mathbb{D})$ is a connected set with $A \nsubseteq U_0(p)$ and $\mathrm{diam}_e A < \hat{r}$, then $\Delta(A) < \pi$. Now, by further replacing $\mathcal{U}_0$ with $\mathcal{U}_n$ (while keeping $U_0(p)$ as part of the cover), we can also assume

$$\mathrm{diam}_e(U_n) < \hat{r} \tag{4}$$

for any $n \geq 0$ and any $U_n \in \mathcal{U}_n$ which is not a pullback of $U_0(p)$. Finally, by removing any redundant open sets in $\mathcal{U}_0$, we may further assume that there are no other elements of $\mathcal{U}_0$ which are contained in $U_0(p)$.

The cover and Frink's lemma. Denote λ^{-N_0} by $r_\star$. Let $\widetilde{\mathcal{U}}_0$ be the lift of $\mathcal{U}_0$ to a neighborhood of Y after blowing up a branching fixed point p for f and its grand orbit, to circles. We add to $\widetilde{\mathcal{U}}_0$ two neighborhoods of arcs in the circle $S_p = \mathbb{R}/2\pi\mathbb{Z}$, V_0 and V_1 with $0 \leq r < r_\star$ and $-\pi/4 < \theta < \pi + \pi/4$ and $\pi - \pi/4 < \theta < 2\pi + \pi/4$ respectively, in the polar coordinates of Lemma 5.2. We replace by them the lift of the set $U_0(p) \in \mathcal{U}_0$. Denote this cover by $\widetilde{\mathcal{W}}_0$, and for each n by $\widetilde{\mathcal{W}}_n$ the cover given by connected components of the sets $g^{-n}(U)$ for any $U \in \widetilde{\mathcal{W}}_0$.

Remark 3. Notice that $\bigcup_n \widetilde{\mathcal{W}}_n$ provides a basis for the topology at all points of Y. Indeed, consider any open $V \ni x$ where $x \in Y$. If $\pi(x) \notin \mathcal{E}$ then by definition V contains $V' = \pi^{-1}(U)$ where U is open in W_0. By [Expansion] for f, U contains $U' \in \mathcal{U}_n$ for some n (see Lemma 2.8), hence $\pi^{-1}(U') \subset V$ and belongs to $\widetilde{\mathcal{U}}_n$. Notice that we can replace the existing n above by all $n \geq N$ for some $N = N_V$. Analogously for $x \in S_q$ we easily choose $U \subseteq g^{-n}(V_i)$ with $x \in U \subseteq V$. □

Given a set $A \subseteq \widetilde{W}_0$, let us denote $A^{(2)} := (A \cap Y) \times (A \cap Y)$, and if $\mathcal{A}$ is a collection of sets, we denote $\mathcal{A}^{(2)} := \bigcup_{A \in \mathcal{A}} A^{(2)}$. Moreover, if $\mathcal{A}, \mathcal{B}$ are collections of subsets of $\widetilde{W}_0$, let us denote their "composition" as

$$\mathcal{A}^{(2)} \circ \mathcal{B}^{(2)} := \{(x, y) \in Y \times Y \ : \ \exists z \in Y \text{ s.t. } (x, z) \in \mathcal{A}^{(2)}, (z, y) \in \mathcal{B}^{(2)}\}.$$

Lemma 5.3. *For the cover $\widetilde{\mathcal{W}}_0$ there exists N such that the following holds:*

(a) *There exists N such that for any $n \geq 0$*

$$\widetilde{\mathcal{W}}_{n+N}^{(2)} \circ \widetilde{\mathcal{W}}_{n+N}^{(2)} \circ \widetilde{\mathcal{W}}_{n+N}^{(2)} \subseteq \widetilde{\mathcal{W}}_n^{(2)}.$$

(b) *If $\mathbf{\Delta} = \{(x, x) \in Y \times Y\}$ is the diagonal, then we have the intersection*

$$\bigcap_{n=0}^{\infty} \widetilde{\mathcal{W}}_{nN}^{(2)} = \mathbf{\Delta}.$$

As a consequence, for the sequence $\Omega_n := \widetilde{\mathcal{W}}_{nN}^{(2)}$ for $n \geq 1$, with $\Omega_0 := Y \times Y$, the hypotheses of Frink's lemma are satisfied.

Proof. (a) We start by proving the claim for $n = 0$; the general case will follow by taking preimages. Consider $W(i) \in \widetilde{\mathcal{W}}_N$, $i = 1, 2, 3$ such that $W(i)$ intersects $W(i+1)$ in Y, for $i = 1, 2$. We prove that if N is large enough then there is $W_0 \in \widetilde{\mathcal{W}}_0$ containing all three $W(i)$.

Let κ_e be the Lebesgue number for $\mathcal{U}_0 \cap X$ and the metric ρ_e^X. Choose N so that $\operatorname{diam}_e \pi(W(i)) \leq \frac{1}{3}\kappa_e$ for $i = 1, 2, 3$. Additionally we assume that N is so large that $\operatorname{diam}_e \pi(W(i)) < r'$ and $\operatorname{diam}_{e'} \pi(W(i) \cap U_0(p)) < r'$ for a constant $r' \ll r_\star$.

Consider $W := \bigcup_{i=1,2,3} \pi(W(i))$. By definition of κ_e, the set $\pi(W) \cap X$ is contained in some element of $\mathcal{U}_0$.

If W lies in one $U_0 \in \mathcal{U}_0$ different from $U_0(p)$, then we are done when we lift these objects to a neighborhood of Y.

So suppose W lies in $U_0(p)$. Fix $r' \ll r_1 \ll r_\star$. We consider two cases.

1. $W \cap X$ is not contained in $h(B(0, r_1))$. Then since its diameter in $\rho_{e'}$ is at most r', all $W(i)$ are contained either in V_0 or in V_1.

2. $W \cap X \subset h(B(0, r_1))$. Let $k > 0$ be the largest integer such that $f^j(W) \subset U_0(p)$ for all $j = 0, \ldots, k$. Since $r_1 \ll r_\star$, the number k is large (note that we do not intersect here W with X). There are two sub-cases.

2A. If $W(i)$ is a pullback of some U in $\mathcal{U}_0 \setminus \{U_0(p)\}$, then $k < N$, no other elements of $\mathcal{U}_0$ are contained in $U_0(p)$. Then, since $f^{k+1}(\pi(W(i)))$ belongs to $\mathcal{U}_{N-k-1}$ and by (4), we have

$$\Delta(f^{k+1}(\pi(W(i)))) < \pi.$$

In the case $g^k(W(i)) = V_0$ or V_1, we obviously also have $\Delta(g^k(W(i))) < \pi$.

2B. Otherwise, $W(i)$ is a pullback of either V_0 or V_1 for $g^{-\ell}$ with $\ell > 0$. Then $\ell > N_0$ and by (3) the set $f^{k+N_0}(W)$ lies in the annulus

$$h(\{\lambda^{-2} \leq r \leq 1\})$$

hence by (3) we obtain $\Delta f^{k+N_0}(W) < \pi$.

Our k is large, so $\Delta(W) \leq \pi d^{-k}$, smaller than $\pi/2$. In 2B. we have used the fact that each $\pi(W(i))$ is a pullback of $f^{k+N_0}(\pi(W(i))) \subset h(\overline{\mathbb{D}})$ for $f^{-(k+N_0)}$. (We would have troubles if we intersected with X without invoking full pullbacks.) Hence all three $W(i) \cap X$ are entirely in V_0 or in V_1. This ends the proof of the claim for the case $n = 0$.

Finally, by taking preimages $\widetilde{\mathcal{W}}_{n+N} = g^{-n}(\widetilde{\mathcal{W}}_N)$ in $\widetilde{\mathcal{W}}_n$ we obtain (a) for all $n \geq 0$.

(b) It follows from the Hausdorff property of the topology on $\widetilde{W}_0$ and the fact that the covers $\widetilde{\mathcal{W}}_{nN}$ are a basis of the topology at Y, as established in Lemma 2.8 and Remark 3.

□

Now, we can apply Frink's lemma (Lemma 2.13) to the sets $\Omega_0 = Y \times Y$ and $\Omega_n := \widetilde{\mathcal{W}}_{nN}^{(2)}$, for $n \geq 1$, obtaining a metric ρ' on Y, which is compatible with the topology introduced above, such that

$$\operatorname{mesh}_{\rho'}\left(\widetilde{\mathcal{W}}_{nN}\right) \leq 2^{-n}$$

for any $n \geq 0$. This shows that the metric ρ' is exponentially contracting with respect to the cover $\widetilde{\mathcal{W}}_N$.

Remark 4. If there are more than one periodic critical point, with period possibly higher than 1, one extends the cover $\widetilde{\mathcal{U}}_0 = \pi^{-1}(\mathcal{U}_0)$ by adding two sets similar to V_0, V_1 for each $p \in \mathcal{E}^*$, and checks the hypotheses of Frink's lemma by applying the previous proof to f^m, where m is the least common multiple of all periods of elements in $\mathcal{E}^*$.

The [Expansion] axiom. Finally, we prove that the cover $\widetilde{\mathcal{W}}_0$ satisfies [Expansion].

Let $\mathcal{V}$ be an arbitrary finite cover of Y. By Remark 3, there exists a cover of Y by the sets $U(x,n) \in \widetilde{\mathcal{W}}_n$, each being a subset of an element V of $\mathcal{V}$ for $n \geq N(V,x)$. Then, by [11, Theorem 3.1.6], together with compactness and the Hausdorff property, we can find, for each $V \in \mathcal{V}$, a compact set $V^* \subset V \cap Y$ so that the union of the V^* covers Y.

Let us now fix $V \in \mathcal{V}$. We shall prove that over all $x \in V^*$, the values $N(V,x)$ are uniformly bounded. Let $V^c := \overline{\widetilde{W}_1} \setminus V$. Recall that for each $q \in \mathcal{E}$ we denote as $m(q)$ the smallest integer $m \geq 1$ such that $f^m(q)$ is an element of $\mathcal{E}^*$.

Case 1. Suppose that $\pi(V^*) \cap \pi(V^c) = \emptyset$. Since π is continuous, these sets are compact, so their Euclidean distance in S^2 is positive, say $\delta_V > 0$. So if $U \in \widetilde{\mathcal{U}}_n$ intersects both V^* and V^c, the Euclidean diameter of $\pi(U)$ is at least δ_V, so by (2) n is bounded from above by some N_V, independent of $x \in V^*$. In other words $U(x,n)$ cannot intersect both V^* and V^c for $n > N_V$, hence is in V.

The same holds for components of $g^{-n}(V_i), i = 1,2$, since their projections by π are in $U_n \in \mathcal{U}_n$ by construction, with diameters tending to 0 by (2).

Case 2. If $\pi(V^*)$ and $\pi(V^c)$ intersect, then there exists $q \in \mathcal{E}$ such that both V^* and V^c intersect S_q. In such a case we have the:

Claim. Among all q's so that V^* and V^c intersect S_q, the set of possible $m(q)$ is bounded.

Proof of the Claim. Otherwise, let $(q_n) \subseteq X$ be a sequence with $m(q_n) \to \infty$, and choose a convergent subsequence $q_n \to q^*$ using the metric ρ_1 on X. Let $x_n \in S_{q_n} \cap V^*$ and $y_n \in S_{q_n} \cap V^c$. If $q^* \notin \mathcal{E}$ then x_n, y_n converge as $n \to \infty$ to the only preimage $\widetilde{q}^*$ of q^* for π because they enter a basis of neighborhoods of $\widetilde{q}^*$, which is the preimage of a basis at q^*. So V^* and V^c intersect, a contradiction. If $q^* \in \mathcal{E}$ then let U be an open neghborhood of q^* given by Remark 3, and let π_{q^*} be the map from $\pi^{-1}(U)$ in $\widetilde{W}_0$ to $\hat{U} := (U \setminus \{q^*\}) \sqcup S_{q^*}$ which blows down every circle except S_{q^*}. The topology and metric on $\hat{U}$ are pullbacks under $f^{m(q^*)}$ of the Euclidean metric and topology from the chart h on a neighborhood of $p = f^{m(q^*)}(q^*)$, extended to the boundary circles. Next choose from $\pi_{q^*}(x_n) = \pi_{q^*}(y_n)$ a subsequence convergent to a point x in S_{q^*}. By definition π_{q^*}

is continuous, so $\pi_{q^*}(V^*)$ and $\pi_{q^*}(V^c)$ are compact, hence they intersect at x. But $\pi_{q^*}^{-1}(x)$ is one point where therefore V^* and V^c intersect, a contradiction. □

Since $m(q)$ is bounded, we need to consider only a finite number of q's. In each S_q, V^* and V^c are compact disjoint so for n large the components of $g^{-n}(V_i)$ in S_q have diameters too small to intersect both V^* and V^c.

The conclusion is that $\widetilde{\mathcal{W}_n}$ is subordinated to $\mathcal{V}$ for all n large enough. The [Expansion] axiom has been verified, which completes the proof of Proposition 5.1.

□

Let us remark that the above proof of [Expansion] together with Theorem 2.12 yields another proof of the existence of an exponentially contracting metric ρ' on Y.

5.1. **Topologically Hölder functions.** Given a cover $\mathcal{V}$ of a topological space X and a function $f : X \to \mathbb{R}$, we define the *total variation* of f with respect to $\mathcal{V}$ as

$$\mathrm{var}_{\mathcal{V}} f := \sup_{V\in\mathcal{V}} \sup_{x,y\in V} |f(x) - f(y)|.$$

Moreover, given a sequence $(\mathcal{V}_n)_{n\geq 0}$ of covers of a topological space X, a function $f : X \to \mathbb{R}$ is *topologically Hölder* with respect to $(\mathcal{V}_n)$ if there exist constants $C, \alpha > 0$ such that

$$\mathrm{var}_{\mathcal{V}_n} f \leq Ce^{-n\alpha}$$

for any $n \geq 0$.

The definition is inspired by Bowen's definition of Hölder continuous potentials for the shift map (see [5, Theorem 1.2]). Let us collect a few fundamental properties about these functions, whose proofs are straightforward.

Lemma 5.4. *Topologically Hölder functions satisfy the following properties.*

(1) Let $f : W_1 \to W_0$ be a finite branched cover with repellor X, let $\mathcal{U}$ be a cover of W_1, and let ρ be an exponentially contracting metric on X. Then a function $\varphi : X \to \mathbb{R}$ which is Hölder continuous with respect to ρ is topologically Hölder with respect to the sequence of covers $\mathcal{U}_n$, whose elements are the connected components of $f^{-n}(\mathcal{U})$.

(2) If $\varphi : X \to \mathbb{R}$ is topologically Hölder with respect to $(\mathcal{U}_n)$ and $\pi : Y \to X$ is continuous, then $\varphi \circ \pi$ is topologically Hölder with respect to $(\mathcal{V}_n) := (\pi^{-1}(\mathcal{U}_n))$.

(3) If $X = \Sigma$ is the full shift space and $(\mathcal{U}_n)$ are the partitions in cylinders of rank n, then $\varphi : X \to \mathbb{R}$ is topologically Hölder with respect to $(\mathcal{U}_n)$ if and only if it is Hölder continuous with the respect to the standard metric on Σ.

Proposition 5.5. *Let $f : W_1 \to W_0$ be a weakly coarse expanding system, with $W_0 \subseteq S^2$, and let (X, ρ) be its repellor, with ρ an exponentially contracting metric. Then for any Hölder continuous potential $\varphi : (X, \rho) \to \mathbb{R}$ there exists a unique equilibrium state μ_X on X.*

Proof. Let $\varphi : X \to \mathbb{R}$ be a Hölder continuous potential with respect to an exponentially contracting metric ρ. By Proposition 5.1, there is a metric space (Y, ρ') with a continuous map $g : Y \to Y$ such that ρ' is exponentially contracting, and a continuous projection $\pi : Y \to X$ such that $\pi \circ g = f \circ \pi$. Moreover, by Proposition 2.16, there exists a coding map $\Pi : \Sigma \to Y$ with $\Pi \circ \sigma = g \circ \Pi$.

By Lemma 5.4(1), the map $\varphi : X \to \mathbb{R}$ is topologically Hölder. Then, by Lemma 5.4 (2), the map $\varphi \circ \pi \circ \Pi$ is also topologically Hölder, hence by Lemma 5.4 (3) the map $\varphi \circ \pi \circ \Pi$ is also Hölder continuous with respect to the standard metric on the shift space Σ.

Thus, by Bowen [5, Theorem 1.22], there is a unique equilibrium state μ on Σ for the potential $\varphi \circ \pi \circ \Pi$. This measure is ergodic with respect to the shift map, and positive on non-empty open sets (as a consequence of the Gibbs property [5, Theorem 1.2]).

Notice that $\Pi_*(\mu)$ is 0 on S_p and its g^n-pre-images. Otherwise, by ergodicity it would be supported on S_p (if it charged other preimages of S_p it would charge preimages under iterates, therefore being infinite). So μ would be 0 on the open nonempty set $\Sigma \setminus \Pi^{-1}(S_p)$, a contradiction.

So $\pi \circ \Pi$ preserves the entropy of μ since Π does (Lemma 3.3) and π does as it is a measurable isomorphism $\pi : Y \setminus \bigsqcup_{p\in\mathcal{E}} S_p \to X \setminus \mathcal{E}$. Moreover, if we set $\mu_X := (\pi\circ\Pi)_*(\mu)$, then $\int_\Sigma \varphi\circ\pi\circ\Pi \, d\mu = \int_X \varphi \, d\mu_X$ by definition.

Hence the projection preserves pressure; namely, by the variational principle

$$P_{top}(f, \varphi) \geq h_{\mu_X}(f) + \int_X \varphi \, d\mu_X = h_\mu(\sigma) + \int_\Sigma \varphi\circ\pi\circ\Pi \, d\mu = P_{top}(\sigma, \varphi\circ\pi\circ\Pi).$$

Since the opposite inequality is true in general, following from definitions, this yields the equality $P_{top}(f, \varphi) = P_{top}(\sigma, \varphi\circ\pi\circ\Pi)$. In particular, we have proved that μ_X is an equilibrium state.

This equilibrium state is unique; indeed, any other equilibrium state would have by Lemma 4.1 a lift to Σ, and this would also be an equilibrium state, as the lift cannot decrease either the entropy nor the integral, contradicting the uniqueness on Σ. □

Proof of Theorem 1.2 (1)-(5). By Proposition 5.5, for any Hölder continuous potential $\varphi : X \to \mathbb{R}$ there is a unique equilibrium state μ_X on X, and this equilibrium state is obtained as the pushforward of the unique equilibrium state ν for the potential $\varphi\circ\pi\circ\Pi$, where $\pi : Y \to X$ is the blowdown map and $\Pi : \Sigma \to Y$ is the coding map. Then, as in the case without periodic critical points, for any Hölder continuous observable ψ statistical laws for the sequence $(\psi \circ f^n)_{n\in\mathbb{N}}$ follow from the well-known statistical laws for the shift map with respect to the sequence $(\psi\circ\pi\circ\Pi\circ\sigma^n)_{n\in\mathbb{N}}$. □

6. The cohomological equation

Let us now see the proof of (6) and (7) in Theorems 1.1 and 1.2.

Proposition 6.1. *Let ρ be a visual metric on X. If the measures μ_φ, μ_ψ coincide then there exist a Hölder continuous function $u : X \to \mathbb{R}$ and a constant $K \in \mathbb{R}$ such that*

$$\varphi - \psi = u\circ f - u + K. \tag{5}$$

Assume that $\mu_\varphi = \mu_\psi = \mu$. We know that $\mu_\varphi = \pi_*\nu_{\varphi\circ\pi}$ and $\mu_\psi = \pi_*\nu_{\psi\circ\pi}$ where $\nu_{\varphi\circ\pi}$, $\nu_{\psi\circ\pi}$ are, respectively, the equilibrium states for $\varphi\circ\pi$ and $\psi\circ\pi$. Put $\eta = \varphi - \psi$. Adding constants to φ, ψ, we may assume that $\int \varphi \, d\mu = 0$ and $\int \psi \, d\mu = 0$, so that $\int \eta \, d\mu = 0$. Denote by $S_k\eta(x)$ the sum

$$S_k\eta(x) = \eta(x) + \cdots + \eta(f^{k-1}(x)).$$

We can detect equality between invariant measures by looking at periodic points:

Proposition 6.2. *If for Hölder continuous $\varphi, \psi : X \to \mathbb{R}$ the equilibrium states satisfy $\mu_\varphi = \mu_\psi$ (denote them by μ) and if $\int \eta \, d\mu = 0$ for $\eta = \psi - \varphi$, then there exists $C > 0$ such that*

$$|S_n\eta(x)| < C \tag{6}$$

for all $x \in X$ and $n \in \mathbb{N}$.

Proof of Proposition 6.2. If μ_φ and μ_ψ coincide, then also $\nu_{\varphi\circ\pi}$ and $\nu_{\psi\circ\pi}$ coincide. Indeed, the proof of Lemma 4.3 shows that any lift of μ_φ is an equilibrium state for $\varphi\circ\pi$. Hence, $\nu_{\psi\circ\pi}$ is an equilibrium state for $\varphi\circ\pi$, but equilibrium states are unique on the shift space, hence $\nu_{\psi\circ\pi} = \nu_{\varphi\circ\pi}$. But then (see, e.g. [5], Theorem 1.28) there exists a constant C such that for every $\omega \in \Sigma$ and every n,

$$|S_n(\eta\circ\pi)(\omega)| \leq C.$$

Since the coding π is onto, this also implies (6). □

Note that $\pi^{-1}(p)$ is not necessarily finite, so it may not be a periodic orbit of σ. So, we conclude the following:

Corollary 6.3. *If μ_φ, μ_ψ coincide and the integral $\int \eta d\mu = \int(\varphi-\psi)d\mu = 0$ then for every periodic point p with $f^k(p) = p$ we have that*

$$S_k\eta(p) = 0. \tag{7}$$

Proof. Suppose $S_k\eta(p) \neq 0$. Then for any n

$$S_{nk}\eta(p) = n \cdot S_k\eta(p),$$

so that the sequence $(S_{nk}\eta)_{n\geq 0}$ is not bounded, a contradiction. □

Using the above characterization, we can solve the cohomological equation (5).

Proof of Proposition 6.1. We can assume that the integral of $\eta = \varphi - \psi$ is equal to zero. As in the classical proof (see [5], proof of Theorem 1.28), begin with choosing a point $x \in X$ with dense trajectory, and define the function u on the set

$$S := \{f^n(x)\}_{n\geq 0}$$

by putting

$$u(x) := 0, \quad u(f^n(x)) := \eta(x) + \cdots + \eta(f^{n-1}(x)) \quad \text{for} \quad n > 0.$$

Then, clearly, for every $z \in S$, $z = f^n(x)$ the equation

$$u(f(z)) - u(z) = \eta(z) = \varphi(z) - \psi(z) \tag{8}$$

holds.

Now, it is enough to prove that the function $u : S \to \mathbb{R}$ is Hölder continuous, i.e., that there exist $C > 0$, $\alpha > 0$ such that

$$|u(f^k(x)) - u(f^m(x))| < C\rho(f^m(x), f^k(x))^\alpha. \tag{9}$$

Then u extends to a Hölder continuous function on X and the equation (8) extends to X.

To prove that u is Hölder continuous, consider two points $z = f^k(x), y = f^m(x)$, $k < m$. Let n be the largest integer such that y, z are in the same component of $\mathcal{U}_n$. Denote this component by U_n.

Let $l := m - k$ so $y = f^l(z)$, and for $r = 1, \ldots l$ denote, by induction by U_{r+n} the connected component of $f^{-1}(U_{r-1+n})$ containing the point $f^{m-r}(x)$. To conclude the proof, we need the following

Lemma 6.4. *There exists a periodic point p in U_{l+n}, of period l, such that $f^{l-r}(p) \in U_{r+n}$, $r = 0, 1, \ldots, l$, and*

$$|S_l\eta(p) - S_l\eta(z)| < C\rho(z, f^l(z))^\alpha. \tag{10}$$

From Lemma 6.4 the estimate (9) follows easily. We have:

$$\begin{aligned} u(f^m(x)) - u(f^k(x)) &= \eta(f^k(x)) + \cdots + \eta(f^{m-1}(x)) \\ &= S_l\eta(z) - S_l\eta(p) \end{aligned}$$

since $S_l\eta(p) = 0$ by Corollary 6.3. By Lemma 6.4,

$$|S_l\eta(p) - S_l\eta(z)| < C\rho(z, f^l(z))^\alpha,$$

so finally

$$|u(f^m(x)) - u(f^k(x))| < C\rho(f^k(x), f^m(x))^\alpha$$

and (9) holds, completing the proof of Proposition 6.1. □

Proof of Lemma 6.4. We use the fact that the map f uniformly expands balls with respect to the visual metric. Indeed, let us fix $\epsilon > 0$, and let ρ be its corresponding visual metric. Then, by [15, Theorem 3.2.1] and [15, Proposition 3.2.2], there exists $r_0 > 0$ such that for any $r < r_0$ and for any $\xi \in X$ one has $f(B(\xi, re^{-\epsilon})) = B(f(\xi), r)$ (in the language of [15], $r_0 = |F(\xi)|_\epsilon$, and that is constant for any $\xi \in \partial_\epsilon \Gamma$, which is our repellor X).

Denote $y = f^l(z)$ and assume $\rho(y, z) < r$ sufficiently small so that we can apply [15, Proposition 3.2.2]. Denote $y_s = f^{m-s}(x)$. Choose consecutive preimages z_s of z such that $\rho(y_s, z_s) \leq e^{-\epsilon}\rho(y_{s-1}, z_{s-1})$, for $s = 0, 1, \dots, l = m - k$. Thus, $\rho(z, z_l) \leq e^{-l\epsilon}\rho(y, z)$.

(Notice that we do not choose just any z_s being a preimage of z_{s-1} in the pullback U_s of $B_{s-1} := B(y_{s-1}, re^{-(s-1)\epsilon})$ containing y_s because U_s can be strictly bigger than B_s. We choose $z_s \in B_s$.)

Now, continue choosing preimages (z_s) for $s \geq l$ close to the previously chosen z_s. This way, we obtain $\rho(z_s, z_{s+l}) \leq e^{-s\epsilon}\rho(z, z_l)$ for any $s \geq 0$, hence $p := \lim_{n\to\infty} z_{nl}$ is the required periodic point.

Indeed we have, for any h with $0 \leq h < l$,

$$\begin{aligned}
\rho(f^h(p), f^h(z)) = \lim_{n\to\infty} \rho(z_{nl-h}, y_{l-h}) &\leq \\
&\leq \rho(z_{l-h}, y_{l-h}) + \sum_{n=1}^{\infty} \rho(z_{nl-h}, z_{(n+1)l-h}) \leq \\
&\leq e^{-\epsilon(l-h)}\rho(y, z) + \sum_{n=1}^{\infty} e^{-(nl-h)\epsilon} e^{-l\epsilon}\rho(y, z) = \\
&= \frac{e^{-\epsilon(l-h)}}{1 - e^{-1}}\rho(y, z)
\end{aligned}$$

hence, using the Hölder continuity of η, there exist $K, \beta > 0$ such that

$$|S_l\eta(p) - S_l\eta(z)| \leq \sum_{h=0}^{l-1} K\rho(f^h(p), f^h(z))^\beta \leq K \sum_{h=0}^{l-1} \left(\frac{e^{-\epsilon(l-h)}}{1 - e^{-1}}\rho(y, z)\right)^\beta \leq C\rho(y, z)^\beta$$

with $C := K(1 - e^{-1})^{-\beta}(1 - e^{-\beta\epsilon})^{-1}$. This completes the proof. □

Proposition 6.1 concludes the proof of Theorem 1.1 (7) in the introduction. Indeed, given any exponentially contracting metric ρ on X there exists a visual metric ρ' which induces the same topology, and, by [15, proof of Theorem 3.2.1, page 58], such that the identity map $(X, \rho') \to (X, \rho)$ is Lipschitz.

Then, φ and ψ are also Hölder continuous with respect to ρ', and we can apply Proposition 6.1 to ρ'. Thus, we obtain u which is Hölder continuous with respect to the visual metric, which means it is continuous with respect to the original metric ρ. Note, however, that we do not know anything about the modulus of continuity of u.

A similar argument also yields the proof of Theorem 1.1 (6). If $\sigma = 0$ and we normalize φ so that $\int \varphi \, d\mu = 0$, then by [25, Lemma 1] we have

$$|S_n(\varphi \circ \pi)(\omega)| < C$$

for any $\omega \in \Sigma$ and any $n \geq 0$, hence by surjectivity also

$$|S_n\varphi(x)| < C$$

for any $x \in X$ and any $n \geq 0$. Now, as in the proof of Proposition 6.1 (with $\psi = 0$) there exists $u : X \to \mathbb{R}$ which is Hölder continuous with respect to the visual metric on X such that

$$\varphi = u \circ f - u.$$

Thus, u is continuous with respect to the original metric, completing the proof.

Finally, the proofs of Theorem 1.2 (6) and (7) proceed exactly in the same way, by considering a visual metric on X and running the same argument as in the proof of Proposition 6.2 and Proposition 6.1, just replacing π with $\pi \circ \Pi$ where $\Pi : \Sigma \to Y$ is the coding map, Y is the repellor for the blown up space as in Section 5, and $\pi : Y \to X$ is as constructed in Proposition 5.1.

7. Appendix A. Exponential contraction

Let us now present a direct construction of an exponentially contracting metric on the blown-up repellor Y, alternative to the approach via Frink's lemma (Lemma 5.3).

Let (X, ρ) be the repellor with a metric ρ which is exponentially contracting with respect to a cover $\mathcal{U}$, let Y be the repellor in the blown up space, let $\pi : Y \to X$ be the blowdown map and let $g : Y \to Y$ be the dynamics on Y, so that $\pi \circ g = f \circ \pi$.

Let us assume for simplicity that there is only one periodic critical point p in X, that $f(p) = p$, and there is no other critical point in the grand orbit of p. Let d be the local degree of f at p.

According to Lemma 5.2, let U be a neighborhood of p in X which is homeomorphic to the unit disc and such that $f : U \cap f^{-1}(U) \to U$ is topologically conjugate to $h(re^{i\theta}) = \lambda re^{id\theta}$ for some $\lambda > 1$.

In the blown up space Y, the disc U is replaced by an annulus A which is homeomorphic to $[0, 1) \times \mathbb{R}/\mathbb{Z}$, and the map g is topologically conjugate to $(r, \theta) \mapsto (\lambda r, d\theta)$. Let $p : A \to \mathbb{R}/\mathbb{Z}$ be the projection to the angular coordinate.

Since we know a local model for g on A, we can define in polar coordinates for each $n \geq 0$ the annulus $A_n := \{(r, \theta) \in A \ : \ \lambda^{-n-2} < r < \lambda^{-n}\}$. Then we know that $g(A_{n+1}) = A_n$. Moreover, let $B_n := \{(r, \theta) \in A \ : \ r < \lambda^{-n}\}$.

(Note that all our metrics are only defined on X. When we write $\mathrm{diam}_\rho(A)$, we mean $\mathrm{diam}_\rho(A \cap X)$.) By replacing f with a power, we can assume that

$$\mathrm{diam}_\rho(A_1) < \rho(\partial A_1, Y \setminus A). \tag{11}$$

Now, let us define for $0 \leq j \leq d^{n+1} - 1$ the open sets

$$A_{n,j} := \{(r, \theta) \in A \ : \ \lambda^{-n-2} < r < \lambda^{-n} \text{ and } j/d^{n+1} < \theta/2\pi < (j+2)/d^{n+1}\}.$$

Note that for any $n > 0$ and any j the map g^n is a homeomorphism from $A_{n,j}$ to some $A_{0,r}$ for $0 \leq r \leq d - 1$. Moreover, let $A^* := Y \setminus \overline{B_1}$, which is an open neighborhood of the complement of A in Y.

Let ρ be an exponentially contracting metric on X, with respect to an open cover $\mathcal{U}$. Consider the projection map $\pi : Y \to X$, and let $\rho_1(x, y) := \rho(\pi(x), \pi(y))$ the pullback of the metric ρ on Y.

On each open set $A_{n,j}$ let us define the pseudo-metric (for $x, y \in Y$)

$$\rho_{A_{n,j}}(x, y) := \lambda^{-n} \rho_1(g^n x, g^n y)$$

with $\lambda = d$. Moreover, on A^* define $\rho_{A^*}(x, y) := 0$. Let $\mathcal{A}$ be the cover which is the union of the $A_{n,j}$ and A^*. Now, given x, y, we say an *admissible chain* γ between x, y is a sequence $(x_i)_{i=0}^k$ of points such that $x = x_0$, $y = x_k$ and for any $0 \leq i \leq k - 1$ both points x_i and x_{i+1} lie in the same element α_i of $\mathcal{A}$. We define the *length* of such an admissible chain as

$$L(\gamma) := \sum_{i=0}^{k-1} \rho_{\alpha_i}(x_i, x_{i+1}).$$

Finally, we define the pseudo-metric ρ_A on $Y \setminus S_p$ as

$$\rho_A(x, y) := \inf_{\gamma \in \Gamma_{x,y}} L(\gamma)$$

where $\Gamma_{x,y}$ is the set of all admissible chains between x and y. Finally, extend ρ_A to S_p by taking its completion.

Properties of ρ_A. Since every concatenation of admissible chains is admissible, ρ_A satisfies the triangle inequality.

Moreover, if $x, y \in A_{n,j}$, then by definition

$$\rho_A(x,y) \leq \rho_{A_{n,j}}(x,y). \tag{12}$$

Finally, if $V_1 \subseteq B_2$ is connected and $V_0 = g(V_1) \subseteq B_1$, then

$$\mathrm{diam}_{\rho_A}(V_1) \leq \lambda^{-1}\mathrm{diam}_{\rho_A}(V_0). \tag{13}$$

Proof. Let $x, y \in V_0$ and $x', y' \in V_1$ with $g(x') = x, g(y') = y$. Now, fix $\epsilon > 0$ and consider an admissible chain γ given by $x = x_0, \dots, x_{k-1}, x_k = y$ between x and y, and such that

$$L(\gamma) \leq \rho_A(x,y) + \epsilon.$$

By (11), for ϵ small such a chain lies entirely in A. Hence, by lifting x_i we find a chain $(y_i)_{i=0}^k$ with $g(y_i) = x_i$. Then each pair y_i, y_{i+1} lies in a set $A_{n_i+1,j_i'}$ with $g(A_{n_i+1,j_i'}) = A_{n_i,j_i}$, and by definition

$$\rho_{n_i+1,j_i'}(z,w) = \lambda^{-1}\rho_{n_i,j_i}(g(z),g(w)) \qquad \forall z,w \in A_{n_i+1,j_i'}$$

hence

$$\rho_A(x',y') \leq \sum_{i=0}^{k-1}\rho_{n_i+1,j_i'}(y_i,y_{i+1}) = \lambda^{-1}\sum_{i=0}^{k-1}\rho_{n_i,j_i}(x_i,x_{i+1}) \leq \lambda^{-1}\rho_A(x,y) + \epsilon$$

and the claim follows as $\epsilon \to 0$. □

(Note that (11) is necessary since the shortest admissible chain between x and y might not lie entirely in A, using a "shortcut" through the complement of A, where ρ_A is zero).

Now, for any q in the grand orbit of p, let k be the smallest integer such that $f^k(q) = p$. Let $A(q)$ be the pullback of A by g^k. Define for each k the set $E_k := f^{-k}(p) \setminus f^{-k+1}(p)$, and let $q \in E_k$. If $g^k : A(q) \to A$ is injective, we define the pseudo-metric

$$\rho_{A(q)}(x,y) := \rho_A(g^k x, g^k y)$$

which is supported on $A(q)$, and given by pulling back the metric ρ_A on $A = A(p)$. If $g^k : A(q) \to A$ is not injective, we define

$$\rho_{A(q)}(x,y) := \inf\left(\sum_{i=1}^{r}\rho_A(g^k x_i, g^k x_{i+1})\right)$$

where the infimum is taken over all finite sets $x = x_0, x_1, \dots, x_r = y$ of points such that x_i and x_{i+1} are both contained in a connected, open set over which g^k is injective.

Definition of the metric $\widetilde{\rho}$. We finally define the metric on Y to be

$$\widetilde{\rho}(x,y) := \rho_1(x,y) + \sum_{n\geq 0}\sum_{q\in E_n} c^n \rho_{A(q)}(x,y) \tag{14}$$

where $c < 1$ is a constant which will be determined later.

Let $\mathcal{U}_n$ be the collection of all connected components of preimages $f^{-n}(U)$ for $U \in \mathcal{U}$. Since ρ is exponentially contracting, then there exist $C_1, \alpha_1 > 0$ such that

$$\sup_{U\in\mathcal{U}_n}\mathrm{diam}_\rho(U) \leq C_1 e^{-n\alpha_1}. \tag{15}$$

By substituting $\mathcal{U}$ with $\mathcal{U}_{n_0}$ for some n_0, we can assume $\mathrm{diam}_\rho(U) < r$ for any $U \in \mathcal{U}_n$ and any n.

Choose $k > 0$ such that $\mathrm{diam}_{\rho_1} A_k < r$. Moreover, denote $W_p^{(1)} := N_\epsilon((r,\theta) \in A \ : \ 0 \leq r \leq \lambda^{-k}, 0 \leq \theta \leq 1/2)$ and $W_p^{(2)} := N_\epsilon((r,\theta) \in A \ : \ 0 \leq r \leq \lambda^{-k}, 1/2 \leq \theta \leq 1)$ where N_ϵ denotes the ϵ-neighborhood in the metric $\widetilde{\rho}$ and $\epsilon > 0$ is small.

Let us consider the open cover $\mathcal{V} := \{\pi^{-1}(U) \ : \ U \in \mathcal{U}\} \cup W_p^{(1)} \cup W_p^{(2)}$. We want to prove that $\widetilde{\rho}$ is exponentially contracting with respect to the cover $\mathcal{V}$. Let $\mathcal{V}_n$ denote the set of connected components of the preimages $g^{-n}(V)$ of elements V of $\mathcal{V}$.

Proposition 7.1. *There exist constants $C, \alpha > 0$ such that for any $V \in \mathcal{V}_n$ we have*

$$\mathrm{diam}_{\widetilde{\rho}}(V) \leq Ce^{-n\alpha}.$$

Proof. The proof is given in several steps.

Step 1. First of all, since ρ is exponentially contracting and $\mathcal{V}$ is a refinement of $\pi^{-1}(\mathcal{U})$, we obtain

$$\sup_{V \in \mathcal{V}_n} \mathrm{diam}_{\rho_1}(V) \leq C_1 e^{-n\alpha_1}. \tag{16}$$

Step 2. Now, we prove by induction that there exist $C_2, \alpha_2 > 0$ such that for each $V \in \mathcal{V}_n$,

$$\mathrm{diam}_{\rho_A}(V) \leq C_2 e^{-n\alpha_2} \tag{17}$$

and $p(V \cap A_0) \neq \mathbb{R}/\mathbb{Z}$. The case $n = 0$ is trivial, as one can just choose the appropriate C_2 since the cover $\mathcal{V}$ is finite.

Now, let $V \in \mathcal{V}_n$ and $V' = g(V) \in \mathcal{V}_{n-1}$. There are two cases.

(1) Let us suppose that $V \subseteq B_2$, so that $V' \subseteq B_1$. Then by (13) we obtain

$$\mathrm{diam}_{\rho_A}(V) \leq \lambda^{-1}\mathrm{diam}_{\rho_A}(V') \leq \lambda^{-1} C_2 e^{-(n-1)\alpha_2} \leq C_2 e^{-n\alpha_2}$$

by taking $\alpha_2 < \log \lambda$.

(2) In the other case, V must be disjoint from B_{k_0} with $k_0 = 3$, hence by (12) and (16), for any $k \leq k_0$ we have

$$\mathrm{diam}_{\rho_A}(V \cap A_{k,j}) \leq \mathrm{diam}_{\rho_{k,j}}(V \cap A_{k,j}) =$$
$$= \lambda^{-k}\mathrm{diam}_{\rho_1}(g^k(V) \cap A_{0,r}) \leq \lambda^{-k}\mathrm{diam}_{\rho_1}(g^k(V)) \leq \lambda^{-k} C_1 e^{-(n-k)\alpha_1}$$

hence, by considering all $A_{k,j}$ for $k \leq k_0$,

$$\mathrm{diam}_{\rho_A}(V) \leq d^{k_0}\lambda^{-k} C_1 e^{-(n-k)\alpha_1}$$

which is less than $C_2 e^{-n\alpha_2}$ if one takes $\alpha_2 < \alpha_1$.

Step 3. Now, let us consider all other contributions to the sum in (14), i.e. the terms relative to $\rho_{A(q)}$ for $q \neq p$. Consider $V \in \mathcal{V}_n$, and let $V_0, V_1, \dots, V_{n-1}, V_n = V$ be a chain of open sets with $V_k \in \mathcal{V}_k$ and V_k the connected component of $g^{-1}(V_{k-1})$.

Consider $q \in E_k$. Then, since the degree of g^k is at most d^k, where d is the global degree of f, by definition of $\rho_{A(q)}$ we have

$$\mathrm{diam}_{\rho_{A(q)}}(V) \leq d^k \mathrm{diam}_{\rho_A}(V_{n-k}) \leq C_2 d^k e^{-(n-k)\alpha_2}$$

Note moreover that the set $f^{-k}(p)$ contains at most d^k elements. Thus, we have for $k \leq n$,

$$\sum_{k=0}^{n} \sum_{q \in E_n} c^k \mathrm{diam}_{\rho_{A(q)}}(V) \leq \sum_{k=0}^{n} c^k d^{2k} C_2 e^{-(n-k)\alpha_2} \leq C_2 e^{-n\alpha_2} \sum_{k=0}^{n} (cd^2 e^{\alpha_2})^k \tag{18}$$

which is exponentially bounded if $cd^2 e^{\alpha_2} < 1$.

Finally, let $M := \mathrm{diam}_{\rho_A}(Y)$. Then

$$\sum_{k>n} \sum_{q \in E_n} c^k \mathrm{diam}_{\rho_{A(q)}}(V) \leq \sum_{k>n} c^k d^{2k} M \leq C_4 (cd^2)^n \tag{19}$$

which is also exponentially bounded if $cd^2 < 1$.

The claim now follows by combining (16), (18) and (19), by choosing an appropriate value of $c < 1$. ∎

8. Appendix B. Embedding into the sphere

Proposition 8.1. *The weakly coarse expanding system $g : \widetilde{W}_1 \to \widetilde{W}_0$ in the assertion of Proposition 5.1 can be continuously embedded into the sphere S^2, where Y becomes a repellor for the extended system. More precisely, there exist a continuous embedding $\iota : \widetilde{W}_0 \to S^2$, a continuous map $g' : S^2 \to S^2$ with $g' \circ \iota = \iota \circ g$ and an open set W_0' which contains $\iota(\widetilde{W}_0)$ so that $\iota(Y) = \bigcap_{n \geq 0} (g')^{-n}(W_0')$.*

Proof. For simplicity, let us assume that there is only one periodic critical point, and that it is actually fixed. Let p be the fixed point for f, and note that $D(p) := S^2 \setminus \{p\}$ is homeomorphic to the unit disc $\mathbb{D}$, and, if $U = h(\mathbb{D})$ is the range of the chart constructed in Lemma 5.2, then $U \setminus \{p\}$ is homeomorphic to an annulus. Thus, let us define a homeomorphism $H : \mathbb{D} \to D(p)$, which we can choose so that it maps the annulus $\{z \in \mathbb{D} \ : \ |z| > 1/2\}$ onto $U \setminus \{p\}$.

Now, we replace p by a circle S_p and add an open disc $D'(p)$, which we parameterize by $H' : \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to D'(p)$ and we identify to $D(p)$ by some homeomorphism $\phi : D(p) \to D'(p)$ so that $\phi \circ H(z) = H'(1/\bar{z})$ for any $z \in \mathbb{D}$.

Then, we construct the space

$$\hat{S}_p := D(p) \sqcup S_p \sqcup D'(p)$$

and define a topology on it by the standard neighborhoods of points in $D(p)$ and $D'(p)$ and by

$$V_{\epsilon,\alpha,\beta} := \{(r, \theta) : 1 - \epsilon < r < 1 + \epsilon, \alpha < \theta < \beta\}$$

for any point $(1, \theta) \in S_p$ (note that here S_p is characterized by $r = 1$, differently from Lemma 5.2). Then the polar coordinates from H, H' glue along $|z| = 1$ to yield a homeomorphism $\hat{H}_p : S^2 \to \hat{S}_p$.

We conclude that $D(p) \sqcup S_p$ with the topology defined in Section 5 embeds by $\hat{H}_p^{-1}$ into S^2 with an open hole (open disc) removed.

Similarly, we consecutively blow up the points q in the grand orbit $O(p)$ to open discs, each time embedding the result in S^2. If $m = m(q)$ is the least integer such that $f^m(q) = p$ and f^m has local degree 1 at q, we use the chart $h_q = f^{-m} \circ h$, extended symmetrically to cover $D'(q)$; if the local degree is higher, we apply adequate roots. (See also the compatible system of charts from Section 5, Remark 2).

Consider $S^2 \setminus O^-(p)$, where $O^-(p)$ is the grand orbit of p. We arrange all $q \in O^-(p)$ different from p into a sequence (q_j) and perform a sequence of embeddings of $S^2 \setminus O^-(p)$ into S^2, blowing up q's as above (the order need not be compatible with $m(q)$; we may forget about the dynamics).

We care that the complementary $D'(q_j)$ have diameters in the spherical metric in S^2 quickly shrinking to 0 and that consecutive embeddings differ from the preceding ones on neighbourhoods of q of diameters also quickly shrinking to 0, so that they form a Cauchy sequence. So the limit embedding ι exists and extends to the closure

$$\widetilde{S} = \Big(S^2 \setminus O^-(p)\Big) \sqcup \bigsqcup_{q \in O^-(p)} S_q$$

with image $S^2 \setminus \bigcup_{q \in O(p)} D'(q)$, which is a Sierpiński carpet.

Crucially, it is possible to perform consecutive perturbations of the embedding not changing the embedding close to the previous q_i's, i.e. not moving already constructed $D'(q)$'s.

Notice finally that we can extend g from a neighborhood $\widetilde{W}_0$ of Y in $\widetilde{S}$ to neighbourhoods of $\iota(S_q)$ in the image S^2, using in $D'(p)$ the formula symmetric to the one in Lemma 5.2. We proceed similarly at $q \in f^{-m}(p)$ using the charts h_q. So Y becomes a repeller in S^2.

In fact the extension can be easily defined on all the $D'(q)$'s, so that $D'(p)$ is the immediate basin of attraction to an attracting fixed point and $\bigcup_{q \in O^-(p)} D'(q)$ the full basin of attraction.

□

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Department of Mathematics and Statistics, University of Wisconsin - La Crosse, 1725 State Street, La Crosse, WI 54601, USA
E-mail address: tdas@uwlax.edu

Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-656 Warszawa, Poland
E-mail address: feliksp@impan.pl

Department of Mathematics, University of Toronto, 40 St George St, Toronto, ON M5S 2E4, Canada
E-mail address: tiozzo@math.utoronto.ca

Department of Mathematics, University of North Texas, 1155 Union Circle #311430, Denton, TX 76203-5017, USA
E-mail address: urbanski@unt.edu

Institute of Mathematics, University of Warsaw, Banacha 2, 02-097 Warszawa, Poland
E-mail address: a.zdunik@mimuw.edu.pl